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When evaluating surface integrals of scalar functions, the (absolute value of the) fundamental vector product $\mathbf{r}_u \times \mathbf{r}_v$ played the role of an area expansion factor, and appeared in the formula we use to directly compute surface integrals. In preparation for understanding how to calculate surface integrals of vector fields, we examine the geometric properties of $\mathbf{r}_u \times \mathbf{r}_v$ more closely.

1. The tangent plane to a parametric surface

Recall that the tangent plane to a surface given by an equation $f(x, y, z) = C$ at the point (a, b, c) has the form

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0,$$

and this formula arose from the fact that the gradient ∇f is orthogonal to level surfaces.

Suppose we are given a surface S using a parametric function $\mathbf{r}(u, v) = \langle X(u, v), Y(u, v), Z(u, v)\rangle$, instead of an implicit function $f(x, y, z) = C$. It might be difficult to find a function f which describes S, so we want to be able to compute the tangent plane to S at a point $\mathbf{r}(u_0, v_0)$ directly from $\mathbf{r}$.

This is actually not too hard to do, and in a sense we already know how to do it. Suppose we want to determine an equation for the tangent plane to S at the point $\mathbf{r}(u_0, v_0)$. If we fix $v = v_0$, and let u vary, the function $\mathbf{r}(u, v_0)$, just as a function of u, traces out a curve on S which passes through $\mathbf{r}(u_0, v_0)$. Furthermore, the tangent vector to this function at $\mathbf{r}(u_0, v_0)$ lies on the tangent plane we are interested in. This tangent vector evidently is equal to $\mathbf{r}_u(u_0, v_0)$. In a similar fashion, if we fix $u = u_0$, and let v vary, the tangent vector to this curve at $\mathbf{r}(u_0, v_0)$ is equal to $\mathbf{r}_v(u_0, v_0)$.

In 'most situations', $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ will not be parallel to each other, so their cross product (which is the fundamental vector product) will be nonzero and orthogonal to both of them. In particular, this tells us that $\mathbf{r}_u \times \mathbf{r}_v$ is orthogonal to the tangent plane to S at $\mathbf{r}(u_0, v_0)$. With this information, we can easily determine an equation for the tangent plane, since we have a normal vector for the plane, as well as a point which the plane passes through.

Examples.

- Let $\mathbf{r}(u, v) = \langle u, v, u^2+v^2\rangle$ be a parameterization for the paraboloid $z = x^2+y^2$. Recall (from last class) that the fundamental vector product for this parameterization is given by

  $$\mathbf{r}_u \times \mathbf{r}_v = \langle -2u, -2v, 1\rangle.$$

  Therefore, the tangent plane to this paraboloid at $\mathbf{r}(u, v)$ has normal vector $\langle -2u, -2v, 1\rangle$ and passes through the point $(u, v, u^2 + v^2)$. This plane has equation

  $$-2ux - 2vy + z = -u^2 + -v^2.$$

  For example, at $(2, 3, 13)$, this tangent plane is

  $$-4x - 6y + z = -13.$$

Notice that if we used the formula with the gradient of $z - x^2 - y^2$, we get the equation

$$-2x_0(x - x_0) + -2y_0(y - y_0) + (z - z_0) = 0,$$

which at $(2, 3, 13)$ is

$$-4(x - 2) - 6(y - 3) + (z - 13) = 0, \text{or} \; -4x - 6y + z = -13.$$

- (Fill an another example here, not as obvious...)

2. The flux across a plane

Suppose we have a vector field $\mathbf{F}$ in $\mathbb{R}^2$ which describes the speed of motion of a fluid of uniform density 1 in the xy plane. For now, let $\mathbf{F}(x, y) = \langle 2, 0\rangle$, for example, so that the fluid is moving to the right at a rate of 2 units per second. Suppose we place a net, represented by the line segment from $(0, 0)$ to $(0, 1)$, in this fluid. Then we might ask, how much fluid passes through the net per unit time?

Intuitively, it seems clear that the amount of fluid passing through the net should be equal to the length of net times the density times the rate at which the fluid is moving. Indeed, in one second, all the fluid from $x = -2$ to $x = 0$, with $0 \le y \le 1$, will pass through the net.

Now suppose the fluid flows at a constant rate in a uniform direction, but this direction is no longer perpendicular to the net. For example, if $\mathbf{F}(x, y) = \langle 0, 1\rangle$, how much fluid passes through the net? In this case, the fluid flow is vertical, but the net is also vertical (and we assume infinitely thin), so no fluid passes through the net. Evidently the orientation of the motion of the fluid relative to the net impacts how much of the fluid passes through the net.

In the first example, the motion of the fluid is perpendicular to the net, while in the second it is parallel. Another way of rephrasing this is as follows: suppose we take a unit normal vector to the net, say $\mathbf{n}$. For example, we can choose $\mathbf{n} = \langle 1, 0\rangle$ for every point on the net, since the net is vertical. Then in the first example, the fluid flow is parallel to $\mathbf{n}$, while in the latter example, it is orthogonal to $\mathbf{n}$.

Suppose the fluid flow $\mathbf{F}$, which we still assume to be uniform in one direction, is now oriented at an angle of θ from $\mathbf{n}$. Then how much fluid passes through the net in one second? Notice that we can resolve the vector $\mathbf{F}$ into a component parallel to $\mathbf{n}$ and a component orthogonal to $\mathbf{n}$. A formula from vector geometry (see Chapter 13) tells us that the component parallel to $\mathbf{n}$ is given by

$$(\mathbf{F} \cdot \mathbf{n})\mathbf{n}.$$

(We use the fact that $\mathbf{n}$ is of unit length to simplify the appearance of this formula.) The component of the fluid flow which is orthogonal to $\mathbf{n}$ evidently contributes nothing to the amount of fluid passing through the net; only the part of the motion which is parallel to $\mathbf{n}$ will contribute. But we already saw that this is equal to the length of $(\mathbf{F} \cdot \mathbf{n})\mathbf{n}$ times the density times the length of the net. One interpretation of $\mathbf{F} \cdot \mathbf{n}$ is that it is equal to $|\mathbf{F}||\mathbf{n}| \cos\theta = |\mathbf{F}| \cos\theta$, where θ is the angle between $\mathbf{F}$ and $\mathbf{n}$. In other words, the larger the angle between $\mathbf{F}$ and $\mathbf{n}$, the smaller the proportion of fluid which flows through the net.

This calculation explains why, on average, parts of the Earth at higher latitude are colder than parts of the Earth near the equator. At high latitudes, the sun stays low on the horizon (for example, at the north pole, the sun never goes above $23.5°$ degrees over the horizon), so the solar energy of the sun, which can be interpreted as a vector field of uniform magnitude and direction to a first approximation, hits the surface of the Earth at an inclined angle.

The total solar energy absorbed by a patch of land of unit area will then be lower at high latitudes, since $\mathbf{F} \cdot \mathbf{n}$ will be smaller, than at low latitudes.

This same principle works for vector fields $\mathbf{F}$ in $\mathbb{R}^3$, with a two-dimensional net S. We take a unit normal vector to S, say $\mathbf{n}$. Since S is assumed to be part of a plane, $\mathbf{n}$ will be constant over S. Then the quantity

$$(\mathbf{F} \cdot \mathbf{n})A(S),$$

which is equal to the component of $\mathbf{F}$ in the direction of $\mathbf{n}$ times the area of S, is called the *flux* of $\mathbf{F}$ across S. Because $\mathbf{F}, \mathbf{n}$ are assumed to be constant right now, another way of writing this is as a surface integral

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS.$$

Example. Calculate the flux of $\mathbf{F} = \langle 1, 1, 1 \rangle$ across the surface S given by $0 \le x \le 1, 0 \le y \le 2, z = 0$.

We begin by finding a unit normal to S. Since S lies in the xy plane, a unit normal is given by either $\langle 0, 0, 1 \rangle$ or $\langle 0, 0, -1 \rangle$. Let's choose $\mathbf{n} = \langle 0, 0, 1 \rangle$. Then $\mathbf{F} \cdot \mathbf{n} = 1$, and the flux is equal to $1(2) = 2$. If we had chosen $\mathbf{n} = \langle 0, 0, -1 \rangle$, notice that we would have obtained the value -2 instead.

3. Orientation of surfaces

The last example illustrates that we need to be more specific when we want to calculate the flux of a vector field across a surface. At any given point, there are going to be two choices of unit normal vectors to a surface S, which point in opposite directions. If we want to calculate the flux of $\mathbf{F}$ across S, we will need to specify some choice of $\mathbf{n}$ to avoid any ambiguity in the final sign of the answer.

This is analogous to the situation with line integrals, where we need to specify an orientation of a curve C to actually calculate a line integral. If we reverse the orientation of a curve on a line integral, the sign changes, just like how if we reverse the sign of the unit normal vector $\mathbf{n}$ used when calculating the flux, the sign of the flux changes.

Therefore, if we want to calculate the flux of $\mathbf{F}$ across a surface S, we should also specify a choice of unit normal vector $\mathbf{n}$ at every point of S. Furthermore, we want this choice of $\mathbf{n}$ to be continuous: that is, we do not want $\mathbf{n}$ to 'flip' to the opposite direction anywhere on S.

When S is part of a plane, it is clear that there are two possible choices for $\mathbf{n}$, and that choosing one of them over the other gives a continuous choice of $\mathbf{n}$ for every point of S. If S is not part of a plane, then $\mathbf{n}$ varies with S, but we still should be able to choose $\mathbf{n}$ at every point of S in such a way so as to ensure that $\mathbf{n}$ is continuous on S.

For example, if S is a sphere, we can either choose $\mathbf{n}$ to point inwards towards the origin, or outwards away from the origin. We can choose $\mathbf{n}$ to be pointing inwards at every point of S, and this choice gives a continuous choice of $\mathbf{n}$. Alternatively, we can choose $\mathbf{n}$ to be pointing outwards at every point of S and this choice also gives a continuous choice of $\mathbf{n}$. What we do not want to do is choose $\mathbf{n}$ to be pointing inwards at some points of S and outwards at other points of S, since then we would need $\mathbf{n}$ to suddenly flip direction somewhere. A continuous choice of $\mathbf{n}$ on S will be called an *orientation* of S.

If you think about various examples of surfaces, you will probably notice that in each example, there are two possible choices for $\mathbf{n}$ to ensure that $\mathbf{n}$ is continuous on S, that these choices point in opposite directions. It may come as a bit of a surprise that there are surfaces for which it is impossible to choose **n** in such a way so as to ensure that **n** is continuous on S.

The canonical example of such a surface is the famous Mobius strip. This is constructed by taking a strip of paper, twisting one end by $180°$, and then gluing the edges together. If you try to choose **n** for this strip in a continuous fashion, you will find that this is impossible, because if you start at a point and then move in a circle along the strip, you will end up at the same point but on the other side of the strip!

We want to rule out surfaces such as this. Surfaces for which there is no continuous choice of **n** are called *non-orientable*, while surfaces for which such a choice exists are called *orientable*. In practice, every surface we encounter will be orientable. If S happens to be a closed surface – that is, a surface with no boundary, like a sphere – the convention is that the preferred orientation of a closed surface is that which points outwards. This orientation is sometimes referred to as the positive orientation of a closed surface, much like how the counterclockwise orientation of a simple closed curve is called its positive orientation.

In summary, if we want to calculate the flux of a vector field **F** across a surface S, we also need to specify an orientation for S, which amounts to a choice of **n**. There are evidently two orientations for S, pointing in opposite directions.