CLASS 20, 5/11/2011, FOR MATH 13, SPRING 2011

In this class, we've seen two ways of generalizing an integral $\int_a^b f(x)\,dx$. On the one hand, we can integrate over higher dimensional regions which lie in $\mathbb{R}^n$, the simplest case being when we integrate over rectangles or rectangular prisms, and on the other hand, we can integrate over one dimensional curves C with line integrals $\int_C f(x,y)\,ds$ or $\int_C \mathbf{F}\cdot d\mathbf{r}$.

The next topic we will cover is a hybrid of these two ideas. We want to integrate over two dimensional regions which may not be flat regions in the plane: that is, we want to learn how to integrate over surfaces.

1. Surface integrals of scalar functions: parametric equations for surfaces

If we want to generalize the idea of a line integral over a curve C to an integral over a two-dimensional surface, let's start by closely examining the various terms in a line integral and the actual method we use to evaluate a line integral.

When we evaluate a line integral of a scalar function $f(x,y)$ over a curve C, we need to know a parameterization $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \le t \le b$ of C. We use this parameterization in the following formula:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2+y'(t)^2}\,dt.$$

Let us examine the $|\mathbf{r}'(t)|$ term more closely. If we pretend that $\mathbf{r}(t)$ describes the motion of a particle, then $|\mathbf{r}'(t)|$ tells us the speed of the particle. Another interpretation of this term is as a length expansion factor: the segment $[t, t+\Delta t]$, which has length Δt, is transformed by $\mathbf{r}(t)$ to the piece of the curve C from $\mathbf{r}(t)$ to $\mathbf{r}(t+\Delta t)$, and this curve has length approximately $|\mathbf{r}'(t)|\Delta t$.

What is the analogous setup if we replace the curve C with a surface S? A surface is two-dimensional, so if we want to give a parametric description of a surface, we will need to use two variables, which we will generally call u, v. The surface S can be described as the set of points of the form $\mathbf{r}(u,v), (u,v) \in D$, where D is some region in the uv plane, and $\mathbf{r}(u,v)$ some function of u, v which takes values in $\mathbb{R}^3$, say. We could let $\mathbf{r}$ take values in $\mathbb{R}^n$ for $n > 3$, but then we would be looking at surfaces in four or more dimensions and we would not be able to visualize these. As a matter of fact, some of the theory we will develop for surfaces in two dimensions will not have an obvious generalization to higher dimensional surfaces, so this restriction on $\mathbf{r}$ is not just cosmetic.

Examples.

- Let $\mathbf{r}(u,v) = \langle u, v, 1-u-v\rangle, u, v \in \mathbb{R}$. This just describes the plane $x+y+z=1$. To see this, notice that the z coordinate of any point on this surface always satisfies $z = 1-x-y$. The fact that u, v are unrestricted corresponds to the fact that we pick up every point on this plane. If we had restricted u, v to the triangle $u+v \le 1, u \ge 0, v \ge 0$, say, then we would only pick up the points on this plane located in the first octant.
- More generally, suppose we have a surface S given by the graph $z = f(x,y)$ of a function over a domain D in the xy plane. Then this graph always has the parameterization $\mathbf{r}(u,v) = \langle u, v, f(u,v)\rangle, (u,v) \in D$. Of course, there are going to

be many alternate parameterizations for S. This is a generalization of the fact that it is easy to parameterize a curve C if that curve is a piece of the graph of a function $y = f(x)$.

There is nothing special about the z-coordinate in this example. If a surface S is given as the graph $y = f(x, z)$ of a function over a domain D in the xz plane, instead, then we can always use the parameterization $\mathbf{r}(u, v) = \langle u, f(u, v), v\rangle, (u, v) \in D$, and similarly if S is given by $x = f(y, z)$.

- Let $\mathbf{r}(u, v) = \langle \cos u, \sin u, v\rangle, 0 \leq u \leq 2\pi, v \in \mathbb{R}$. This parametric equation describes the cylinder given by the equation $r = 1$. To see this, notice that as u varies from 0 to 2π, the x, y coordinates describe a circle of radius 1 in the xy plane. As v varies, this circle moves up and down in the z direction.

  If we fix u and let v vary, we obtain various vertical lines on the surface of this cylinder, while if we let u vary and fix v, we obtain circles on the surface of the cylinder parallel to the xy plane. These rulings will play an important role when we actually discuss integrating over surfaces.
- Let $\mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle, 0 \leq u \leq \pi, 0 \leq v \leq 2\pi$. It looks like it might be difficult to determine what the image of $\mathbf{r}$ is, but these equations actually have a very familiar form. They look like the equations for spherical coordinates, with $\rho = 1, u = \phi, v = \theta$. Therefore, this parametric equations determines a sphere of radius 1.

2. Surface integrals of scalar functions

Suppose we have a surface S given by a parametric equation $\mathbf{r}(u, v) = \langle X(u, v), Y(u, v), Z(u, v)\rangle, (u, v) \in D$. Suppose we also have a function $f(x, y, z)$ defined on S. We want to define the surface integral of $f(x, y, z)$ on S in such a way so that we simultaneously generalize the notion of a line integral and double integral.

Recall that for line integrals we use the formula

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\, dt,$$

and we can interpret $|\mathbf{r}'(t)|$ as a length magnification factor. This suggests that we should attempt to determine an area magnification factor for the parametric equation $\mathbf{r}(u, v)$ which defines S. More precisely, we want to find a function of u, v which tells us how much a small rectangle of side lengths $\Delta u, \Delta v$, with coordinates about equal to (u, v), is magnified under $\mathbf{r}$.

The image of this small rectangle under $\mathbf{r}$ will be a small patch of S bounded by the various rulings we looked at for some of the examples of the previous section. This patch can be approximated by a parallelogram with sides of length

$$|\mathbf{r}_u(u, v)|\Delta u, |\mathbf{r}_v(u, v)|\Delta v,$$

where $\mathbf{r}_u, \mathbf{r}_v$ indicate the vector-valued functions we obtain by taking partial derivatives of the component functions of $\mathbf{r}$ with respect to either u or v. That is,

$$\mathbf{r}_u(u, v) = \left\langle \frac{\partial X}{\partial u}(u, v), \frac{\partial Y}{\partial u}(u, v), \frac{\partial Z}{\partial u}(u, v) \right\rangle, \mathbf{r}_v(u, v) = \left\langle \frac{\partial X}{\partial v}(u, v), \frac{\partial Y}{\partial v}(u, v), \frac{\partial Z}{\partial v}(u, v) \right\rangle.$$

Why are these good first-order approximations to the side lengths of this parallelogram? One side of the parallelogram is an approximation to the side of the patch of S which is obtained by fixing v while letting u vary, so its length will be approximated by $|\mathbf{r}_u(u, v)|\Delta u$. In any case, we know how to calculate the area of a parallelogram if its sides are given by two vectors in $\mathbb{R}^3$: we take the absolute value of the cross-product! The small patch of S thus has area approximately equal to

$$|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)|\Delta u \Delta v.$$

Since the original small rectangle had area $\Delta u \Delta v$, $\mathbf{r}(u,v)$ magnifies areas by a factor of

$$|\mathbf{r}_u \times \mathbf{r}_v|.$$

Therefore, it should be no surprise that the *surface integral* of $f(x,y,z)$ over S is given by the equation

$$\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(X(u,v),Y(u,v),Z(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA.$$

The term $\mathbf{r}_u \times \mathbf{r}_v$ is sometimes known as the *fundamental vector product* of the surface S. Just like how we directly evaluate line integrals by reducing them to a usual definite integral over an interval $[a,b]$, we evaluate surface integrals by reducing them to a double integral over a region D lying in $\mathbb{R}^2$. The differential 'dS'' means that we are integrating with respect to the surface area of S.

There are a variety of places where surface integrals appear naturally, both in mathematics and physics. For example, if we have a surface S whose density is given by $\rho(x,y,z)$, then its mass is given by the surface integral of ρ over S. Similarly, there are formulas for determining the center of mass of a surface, its moment of inertia about various axes, etc. A geometric application of surface integrals is that it allows us to calculate surface area of various surfaces. If we integrate the constant function $f(x,y,z) = 1$, we will obtain the surface area of S.

Examples.

- Let S be the part of the plane $z = 1 - x - y$ with $x \geq 0, y \geq 0, z \geq 0$. Evaluate the surface integral of $f(x,y,z) = xy$ over S.

  We use the parameterization we found in the first example of the previous section: $\mathbf{r}(u,v) = \langle u, v, 1-u-v\rangle$. The restrictions on x,y,z correspond to the restrictions $u \geq 0, v \geq 0, u+v \leq 1$, so D is the region of the uv plane given by these inequalities.

  The partial derivatives of this vector-valued function with respect to u,v are given by

  $$\mathbf{r}_u = \langle 1, 0, -1\rangle, \mathbf{r}_v = \langle 0, 1, -1\rangle.$$

  Therefore, the fundamental vector product is given by

  $$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1\rangle.$$

  The surface integral of $f(x,y,z) = xy$ over S is therefore given by

  $$\iint\limits_S xy\,dS = \iint\limits_D uv|\sqrt{3}|\,dA = \int_0^1 \int_0^{1-v} uv\sqrt{3}\,du\,dv$$

  $$= \int_0^1 \frac{v(1-v)^2}{2}\,dv = \int_0^1 \frac{v - 2v^2 + v^3}{2}\,dv = \sqrt{3}\left(\frac{v^2}{4} - \frac{v^3}{3} + \frac{v^4}{8}\Big|_0^1\right) = \frac{\sqrt{3}}{24}.$$

- Use a surface integral to find the surface area of a sphere with radius R.

  A sphere of radius R, centered at the origin, can be described by a parameterization

  $$\mathbf{r}(u,v) = \langle R\sin u\cos v, R\sin u\sin v, R\cos u\rangle, 0 \leq u \leq \pi, 0 \leq v \leq 2\pi,$$

  since this is a description of this sphere using spherical coordinates, with $\rho = R, \phi = u, \theta = v$. We take the partial derivatives of this parameterization with respect to u and v:

  $$\begin{aligned} \mathbf{r}_u &= R\langle\cos u\cos v, \cos u\sin v, -\sin u\rangle \\ \mathbf{r}_v &= R\langle -\sin u\sin v, \sin u\cos v, 0\rangle \end{aligned}$$

  The fundamental vector product is equal to

  $$\mathbf{r}_u \times \mathbf{r}_v = R^2 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u\cos v & \cos u\sin v & -\sin u \\ -\sin u\sin v & \sin u\cos v & 0 \end{vmatrix}.$$

  This looks really complicated, and it is somewhat computationally difficult to simplify, but it turns out that an application of $\cos^2\theta + \sin^2\theta = 1$ gives

  $$\mathbf{r}_u \times \mathbf{r}_v = R^2\langle\sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u\rangle.$$

  This is still a bit complicated, but it turns out that the absolute value of this expression can be simplified a lot by using $\cos^2\theta + \sin^2\theta = 1$, and one gets

  $$|\mathbf{r}_u \times \mathbf{r}_v| = R^2\sqrt{\sin^2 u} = R^2|\sin u| = R^2\sin u,$$

  since $0 \leq u \leq \pi$, so that $\sin u \geq 0$. Therefore, the surface area of the original sphere is given by

  $$\iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_0^{2\pi}\int_0^{\pi} R^2\sin u\, du\, dv = 4\pi R^2.$$

  Notice that the absolute value of the fundamental vector product is basically the Jacobian of the spherical coordinate system. This is not a coincidence, since the Jacobian can be interpreted as a local volume magnification factor.
- Use a surface integral to find the surface area of the paraboloid $z = x^2 + y^2$ lying above the disc $x^2 + y^2 \leq 1$ in the xy plane.

  Again, the easiest parameterization to use for this surface is

  $$\mathbf{r}(u,v) = \langle u, v, u^2 + v^2\rangle.$$

  The partial derivatives of this parameterization with respect to u, v are

  $$\begin{aligned} \mathbf{r}_u(u,v) &= \langle 1, 0, 2u\rangle \\ \mathbf{r}_v(u,v) &= \langle 0, 1, 2v\rangle \end{aligned}$$

  The fundamental vector product is given by

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = \langle -2u, -2v, 1 \rangle,$$

and has length equal to $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{4u^2 + 4v^2 + 1}$. Let D be the disc $u^2 + v^2 \leq 1$. Then the surface area of this paraboloid is equal to

$$\iint\limits_S dS = \iint\limits_D \sqrt{4u^2 + 4v^2 + 1}\, dA.$$

A good strategy for evaluating this double integral is to use polar coordinates. Letting $u = r\cos\theta, v = r\sin\theta$, this double integral is equal to the iterated integral

$$\int_0^{2\pi} \int_0^1 r\sqrt{4r^2+1}\, dr\, d\theta = 2\pi \frac{2(4r^2+1)^{3/2}}{3 \cdot 8} \bigg|_{r=0}^{r=1} = \frac{\pi(4r^2+1)^{3/2}}{6} \bigg|_{r=0}^{r=1} = \frac{(5^{3/2}-1)\pi}{6}.$$

As you can see, calculating surface integrals is a fairly lengthy process and often requires that you draw upon everything you have learned in this class so far. You need to know how to take partial derivatives, take cross products, evaluate lengths of vectors, and then evaluate various types of double integrals, which may involve using polar coordinates or other techniques (such as interchanging the order of integration) we learned in this class. Practice is the best way to become mechanically proficient at evaluating surface integrals!