CLASS 19, 5/9/2011, FOR MATH 13, SPRING 2011

1. Surface integrals of scalar functions: parametric equations for surfaces

In this class, we've seen two ways of generalizing an integral $\int_a^b f(x)\,dx$. On the one hand, we can integrate over higher dimensional regions which lie in $\mathbb{R}^n$, the simplest case being when we integrate over rectangles or rectangular prisms, and on the other hand, we can integrate over one dimensional curves C with line integrals $\int_C f(x,y)\,ds$ or $\int_C \mathbf{F}\cdot d\mathbf{r}$.

The next topic we will cover is a hybrid of these two ideas. We want to integrate over two dimensional regions which may not be flat regions in the plane: that is, we want to learn how to integrate over surfaces.

If we want to generalize the idea of a line integral over a curve C to an integral over a two-dimensional surface, let's start by closely examining the various terms in a line integral and the actual method we use to evaluate a line integral.

When we evaluate a line integral of a scalar function $f(x,y)$ over a curve C, we need to know a parameterization $\mathbf{r}(t) = \langle x(t), y(t)\rangle, a \le t \le b$ of C. We use this parameterization in the following formula:

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Let us examine the $|\mathbf{r}'(t)|$ term more closely. If we pretend that $\mathbf{r}(t)$ describes the motion of a particle, then $|\mathbf{r}'(t)|$ tells us the speed of the particle. Another interpretation of this term is as a length expansion factor: the segment $[t, t+\Delta t]$, which has length Δt, is transformed by $\mathbf{r}(t)$ to the piece of the curve C from $\mathbf{r}(t)$ to $\mathbf{r}(t+\Delta t)$, and this curve has length approximately $|\mathbf{r}'(t)|\Delta t$.

What is the analogous setup if we replace the curve C with a surface S? A surface is two-dimensional, so if we want to give a parametric description of a surface, we will need to use two variables, which we will generally call u, v. The surface S can be described as the set of points of the form $\mathbf{r}(u,v), (u,v) \in D$, where D is some region in the uv plane, and $\mathbf{r}(u,v)$ some function of u, v which takes values in $\mathbb{R}^3$, say. We could let $\mathbf{r}$ take values in $\mathbb{R}^n$ for $n > 3$, but then we would be looking at surfaces in four or more dimensions and we would not be able to visualize these. As a matter of fact, some of the theory we will develop for surfaces in two dimensions will not have an obvious generalization to higher dimensional surfaces, so this restriction on $\mathbf{r}$ is not just cosmetic.

Examples.

- Let $\mathbf{r}(u,v) = \langle u, v, 1-u-v\rangle, u, v \in \mathbb{R}$. This just describes the plane $x+y+z=1$. To see this, notice that the z coordinate of any point on this surface always satisfies $z = 1-x-y$. The fact that u, v are unrestricted corresponds to the fact that we pick up every point on this plane. If we had restricted u, v to the triangle $u+v \le 1, u \ge 0, v \ge 0$, say, then we would only pick up the points on this plane located in the first octant.
- More generally, suppose we have a surface S given by the graph $z = f(x,y)$ of a function over a domain D in the xy plane. Then this graph always has the parameterization $\mathbf{r}(u,v) = \langle u, v, f(u,v)\rangle, (u,v) \in D$. Of course, there are going to

be many alternate parameterizations for S. This is a generalization of the fact that it is easy to parameterize a curve C if that curve is a piece of the graph of a function $y = f(x)$.

There is nothing special about the z-coordinate in this example. If a surface S is given as the graph $y = f(x, z)$ of a function over a domain D in the xz plane, instead, then we can always use the parameterization $\mathbf{r}(u, v) = \langle u, f(u, v), v\rangle, (u, v) \in D$, and similarly if S is given by $x = f(y, z)$.

- Let $\mathbf{r}(u, v) = \langle \cos u, \sin u, v\rangle, 0 \leq u \leq 2\pi, v \in \mathbb{R}$. This parametric equation describes the cylinder given by the equation $r = 1$. To see this, notice that as u varies from 0 to 2π, the x, y coordinates describe a circle of radius 1 in the xy plane. As v varies, this circle moves up and down in the z direction.

  If we fix u and let v vary, we obtain various vertical lines on the surface of this cylinder, while if we let u vary and fix v, we obtain circles on the surface of the cylinder parallel to the xy plane. These rulings will play an important role when we actually discuss integrating over surfaces.
- Let $\mathbf{r}(u, v) = \langle \sin u \cos v, \sin u \sin v, \cos u\rangle, 0 \leq u \leq \pi, 0 \leq v \leq 2\pi$. It looks like it might be difficult to determine what the image of $\mathbf{r}$ is, but these equations actually have a very familiar form. They look like the equations for spherical coordinates, with $\rho = 1, u = \phi, v = \theta$. Therefore, this parametric equations determines a sphere of radius 1.