CLASS 14, 4/27/2011, FOR MATH 13, SPRING 2011

1. Vector fields

We've briefly seen how to calculate line integrals of real-valued functions over various curves C, which can lie either in $\mathbb{R}^2$ or $\mathbb{R}^3$. These integrals were motivated by various real-life applications: for example, a line integral of f over C can be interpreted as the (signed) surface area of a surface lying over C with height f, or as the mass (or total electric charge) of a wire whose density is given by f.

However, a full treatment of the theory of line integrals requires that we learn how to integrate not just real-valued functions, but also vector-valued functions. The most obvious application of this is to the calculation of the work done on a particle by a force. Recall that in high school physics one learns $W = Fd$, so long as the force and the direction an object moves are parallel to each other. After learning about vectors, one learns that $W = \mathbf{F} \cdot \mathbf{d}$, so that we can calculate work even if the force and direction are not parallel to each other. What happens if the direction of motion as well as the force are constantly changing? For example, perhaps a particle moves along a curve C, and the force acting on it varies on C. How much work does the force do to the particle then? This is the type of question line integrals were designed to answer.

Before we actually do any integrals we will briefly discuss vector fields, which are the objects we will integrate. A *vector field* is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, or more generally, a function $\mathbf{F} : D \to \mathbb{R}^n$, where D is a subset of $\mathbb{R}^n$. In this class we will deal almost exclusively with the cases $n = 2, 3$. We think of vector fields as functions which assign a vector to each point of $\mathbb{R}^n$. For example, if $n = 3$, we think of a vector field as being a function which assigns to each point (x, y, z) a vector $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. From now on we will reserve the (capital) letters P, Q, R for the components of a vector field. If we are looking at vector fields on $\mathbb{R}^2$, we just use $P(x, y), Q(x, y)$.

We graphically represent vector fields by plotting the vectors $\mathbf{F}(x, y)$ at a selection of points, with their tails starting at the point (x, y) in question. Such sketches do not contain all the information about $\mathbf{F}$, since we are only plotting some finite number of vectors (contrast this to when you sketch the graph of a function $y = f(x)$, where you plot every value of $f(x)$ on an interval), but often can provide a good intuitive idea of the behavior of the vector field. When drawing vector fields it is important that you get the relative length of various vectors correct, as well as the direction of those vectors.

Examples.

- Plot the vector field $\mathbf{F}(x, y) = \langle 1, 1\rangle$. This is a vector field which assigns the vector $\langle 1, 1\rangle$ to every point of $\mathbb{R}^2$. So a sketch of this vector field looks like a sketch of lots of parallel vectors, each of the same length.
- Plot the vector field $\mathbf{F}(x, y) = \langle x, y\rangle$. This is a vector field whose vectors point *radially outward*; that is, they all point away from the origin. Furthermore, the length of a vector is equal to its distance from the origin, so these vectors get larger and larger the further from the origin we get.
- Plot the vector field $\mathbf{F}(x, y) = \langle y, -x\rangle$. This is a vector field whose vectors are perpendicular to the line segment connecting (x, y) to the origin. To see this, take the dot product of $\langle x, y\rangle$ with $\langle y, -x$. This dot product equals 0, which means these two vectors are orthogonal. If we draw in circles $x^2 + y^2 = r^2$, for various r, we

see that these vectors are tangent to the circles, and that these vectors point in the counterclockwise direction along these circles. The magnitude of these vectors is equal to the distance of (x, y) from the origin, and so increases as we get further out from the origin.

- Plot the vector field $\mathbf{F}(x, y) = \frac{1}{\sqrt{x^2+y^2}}\langle x, y\rangle$. Again, these vectors point radially outward. However, their lengths are all equal to 1, so while the direction of these vectors change with (x, y), their magnitude does not. Notice that this vector field is not defined at $(0, 0)$, and cannot be defined at $(0, 0)$ in any way so as to make the vector field 'continuous'.
- The most common application of vector fields in physics and engineering is to the description of gravitational or electric fields. For example, suppose we place a positive charge of 1 unit at the origin. Suppose we place some other charge, of charge q, at a point (x, y). Then Coulomb's Law says that the force the second charge feels due to the first is given by
$$\frac{q}{d^2}$$
where d is the distance between the two charges, and the direction of the force is pointing away from the origin. If $q > 0$, this corresponds to the fact that two like charges repel, and if $q < 0$, this corresponds to the fact that two opposite charges attract. (This does not look exactly like the expression you might see in physics classes or texts, where some constants can appear. We are selecting our units in such a way to ensure that all relevant constants equal 1.) We can re-express this using vectors by saying that the force is equal to the vector
$$\mathbf{F} = \frac{q}{d^3}\mathbf{d},$$
where $\mathbf{d}$ is the vector $\langle x, y\rangle$; ie, the vector which starts at the origin and ends at the second charge. We define the electric field that the charge at the origin produces to be the field $\mathbf{E}$ defined by $\mathbf{E}q = \mathbf{F}$; that is, given an electric field, we can calculate the force the particle responsible for the field exerts on another particle by simply multiplying the charge of that particle by the field. The electric field generated by our original particle is directed radially outward, with vectors of magnitude $1/d^2$. This explains why vector fields which point radially outward or inward are so important – because they appear in nature. Again, notice that there is no way to define this field at the origin in such a way so as to make the vector field continuous.
- Another physical interpretation of a vector field is as the rate of flow of a fluid. For example, a vector field $\mathbf{F}(x, y) = \langle e^x, 0\rangle$ could represent a fluid which flows exponentially faster as x increases. The vector $\mathbf{F}(x, y)$ represents the amount of fluid which is passing through (x, y) per unit time.

2. Gradients and vector fields

Suppose f is a scalar function on $\mathbb{R}^2$ or $\mathbb{R}^3$. For example, suppose $f(x, y)$ is defined on $\mathbb{R}^2$, and is differentiable everywhere. Then the gradient of $f(x, y)$ is defined by $\nabla f(x, y) = \langle f_x(x, y), fy(x, y)\rangle$. Notice that this is also a vector field!

In general, we call any vector field $\mathbf{F}$ which is also equal to the gradient of some function a *gradient vector field* or *conservative vector field.* If $\mathbf{F}$ is conservative, with $\mathbf{F} = \nabla f$, we sometimes call f a *potential function* for $\mathbf{F}$.

Examples.

- The electric field generated by a particle is conservative. One can check that the function $f(x,y) = -(x^2+y^2)^{-1/2}$ is a potential function for $\mathbf{E}$ by direct calculation. The physical interpretation of $f(x,y)$ is that it represents the potential energy of the electric field at a point (x,y). For example, a particle very far from the origin should have high potential energy, while a particle very close to the origin should have low potential energy, which is exactly what happens with this $f(x,y)$. We will see this interpretation of the potential function in a more precise form next week.
- On the other hand, not every vector field is conservative. For example, let $\mathbf{F}(x,y) = \langle 2y, 3x^2\rangle$. If $\mathbf{F}$ were conservative, say $\mathbf{F} = \nabla f$, then $f_x = 2y, f_y = 3x^2$. These are both continuous functions, and their partial derivatives are also continuous, so by an application of Clairaut's Theorem we would then have $f_{xy} = f_{yx}$. However, $f_{xy} = 2, f_{yx} = 6x$, which are not equal, so there is no way that this function could be conservative.

We will first learn how to integrate vector fields over curves C. We then want to investigate several questions, many of which touch on conservative vector fields. For example, do conservative vector fields have any special properties, especially with regards to line integrals? How can we determine whether or not a vector field is conservative? And what is the origin of the name 'conservative'?

3. Line integrals of vector fields: definition

Now that we've defined vector fields, look at a few examples of vector fields, and also defined the special class of conservative vector fields, let's look at how to evaluate line integrals of vector fields.

The definition of the line integral of a vector field $\mathbf{F}$ along a curve C is similar to the definition of the line integral of a scalar function along C. This definition should allow us to calculate the work a force (possibly varying with position) exerts on a particle as that particle travels in a curve C.

Suppose C is a curve in $\mathbb{R}^2$, parameterized by the vector-valued function $\mathbf{r}(t), a \le t \le b$. If $\mathbf{F}$ is a vector field defined on C, then the *line integral of* $\mathbf{F}$ *along* C is given by the formula

$$\int_C \mathbf{F}\, d\mathbf{r} = \int_a^b \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)\, dt.$$

Notice the similarity of this formula to the analogous formula for the line integral of a scalar function. If we think of $\mathbf{r}(t)$ as describing the position of a particle, recall that $\mathbf{r}'(t)$ describes the instantaneous velocity of that particle. Therefore, if $\mathbf{F}$ is interpreted as a force, then $\mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(t)$ describes the work per second being done on the particle at a given time t, and integrating this expression with respect to t will calculate the total work done over the entire path of the particle.

- Sometimes line integrals of vector fields are written slightly differently than what we mentioned above. If $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a description of $\mathbf{F}$ in terms of its components, then we sometimes write the line integral of $\mathbf{F}$ over C as

  $$\int_C \mathbf{F}\, d\mathbf{s} = \int_C P\, dx + Q\, dy.$$

  The interpretation of this notation is that when calculating, say

  $$\int_C P\, dx,$$

  which looks like the line integral of the scalar function P (the only difference being the presence of dx instead of ds as the differential in the integral sign), when we calculate this integral by expressing this as an integral of a function in t, we should replace dx with $x'(t)\,dt$:

$$\int_C P\,dx = \int_a^b P(x(t), y(t))x'(t)\,dt.$$

Therefore, with this interpretation of these terms (sometimes called the line integral of P along C with respect to x), we have

$$\int_C P\,dx + Q\,dy = \int_a^b P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)\,dt = \int_a^b \mathbf{F}'(x(t), y(t)) \cdot \mathbf{r}'(t)\,dt.$$

Therefore, this different notation for line integrals of vector fields is compatible with the original definition, as it should be.

- When calculating the line integral of a scalar function f along C, we needed to find a parameterization $\mathbf{r}(t)$ for C. However, the parameterization we chose did not matter, as long as $\mathbf{r}(t)$ touched each point of C exactly once. When calculating line integrals of vector fields, the choice of parameterization can affect the value of the line integral. In particular, given a curve C, there are two choices of directions for C, called an *orientation* for C. A parameterization corresponds to a given orientation depending on the direction it traverses as t increases; it turns out that the value of a line integral will change sign if the orientation describing C is reversed. However, if two parameterizations have the same orientation, the value of the line integral will be the same using either of the parameterizations.
- We initially define line integrals over curves C which can be described by a differentiable function $\mathbf{r}(t)$. We extend the definition to curves which are piecewise differentiable – that is, curves which are everywhere differentiable except at a finite number of points – by adding up the value of the line integrals over each of the individual pieces.

Let us look at some examples illustrating the calculation of line integrals over vector fields, as well as some of the remarks above.

Examples.

- Let $\mathbf{F} = \langle y, 2x\rangle$, and let C be the curve which is the line segment from $(0, 0)$ to $(1, 1)$. Suppose C is given by parameterization $\mathbf{r}(t) = \langle t, t\rangle, 0 \le t \le 1$. Calculate the line integral of $\mathbf{F}$ on C using this parameterization.

  We first remark that we can write this line integral in either of the forms

  $$\int_C \mathbf{F} d\mathbf{r} = \int_C y\,dx + 2x\,dy.$$

  We have $\mathbf{r}'(t) = \langle 1, 1\rangle$ and $\mathbf{F}(x(t), y(t)) = \langle t, 2t\rangle$. Therefore, this line integral is equal to

  $$\int_0^1 \langle t, 2t\rangle \cdot \langle 1, 1\rangle\,dt = \int_0^1 3t\,dt = \frac{3}{2}.$$

- Let $\mathbf{F} = \langle y, 2x\rangle$, and let C be the curve which is the line segment from $(0, 0)$ to $(1, 1)$. Suppose C is given by the parameterization $\mathbf{r}(t) = \langle t^2, t^2\rangle, 0 \le t \le 1$. Calculate the line integral of $\mathbf{F}$ along C using this parameterization.

  This time, we have $\mathbf{r}'(t) = \langle 2t, 2t\rangle$. Therefore, the line integral in question is equal to

  $$\int_0^1 \langle t^2, 2t^2\rangle \cdot \langle 2t, 2t\rangle\,dt = \int_0^1 2t^3 + 4t^3\,dt = \frac{6}{4} = \frac{3}{2}.$$

This example illustrates the fact that, even though this example and the previous example have different parameterizations for C, the resulting value of the line integral is the same, since both parameterizations describe the same orientation for C.

- Let $\mathbf{F} = \langle y, 2x \rangle$, and let C be the curve which is the line segment from $(0, 0)$ to $(1, 1)$. (This is the same setup as the previous two examples!) Suppose C is given by parameterization $\mathbf{r}(t) = \langle 1-t, 1-t \rangle, 0 \le t \le 1$. Calculate the line integral of $\mathbf{F}$ on C using this parameterization.

  This example differs from the previous two in that while $\mathbf{F}, C$ are the same, the parameterization for C is a different orientation now. We have $\mathbf{r}'(t) = \langle -1, -1 \rangle$, so the line integral in question equals
  $$\int_0^1 \langle 1-t, 2(1-t) \rangle \cdot \langle -1, -1 \rangle \, dt = \int_0^1 -3(1-t) \, dt = \frac{3(1-t)^2}{2} \Big|_0^1 = \frac{-3}{2}.$$
  The answer is the negative of the answer from the previous two examples, exactly as expected given our remarks earlier.

- Let $\mathbf{F} = \langle y, 2x \rangle$, and let C be the curve given by parameterization $\mathbf{r}(t) = \langle t, t^2 \rangle, 0 \le t \le 1$. Calculate the line integral of $\mathbf{F}$ along C using this parameterization.

  This example differs from the previous three in that C is no longer a line segment, but a piece of a parabola. However, this C has the same endpoints as the C used in the previous examples.

  We have $\mathbf{r}'(t) = \langle 1, 2t \rangle$. Therefore, the line integral of $\mathbf{F}$ along C is
  $$\int_0^1 \langle t^2, 2t \rangle \cdot \langle 1, 2t \rangle \, dt = \int_0^1 5t^2 \, dt = \frac{5}{3}.$$

Notice that this value differs from the previous three examples. This should not be too surprising since there is no a priori reason to believe that the value of a line integral should depend only on the endpoints of C. However, there is a special class of vector fields for which the values of line integrals along various paths C do not depend on C, and only depend on the endpoints of C.