CLASS 13, 4/25/2011, FOR MATH 13, SPRING 2011

1. Line integrals

We have spent a few weeks talking about higher-dimensional generalizations of definite integrals, and discussed how to calculate them, as well as various applications of double and triple integrals in real life. We now try to generalize the notion of an integral in a different direction. Instead of focusing on defining integrals over high dimensional objects, we will develop a theory of how to integrate not over just an interval on the real axis, but over curves.

Let's motivate this idea with a whimsical example. The American artist Richard Serra is known for his minimalist sculptures, many of which are curving sheets of metal. If you were working for Richard Serra a question you might encounter is just how much sheet metal you need to make one of his installations. In more mathematically precise language, suppose an installation of sheet metal is going to be placed over a curve C in the xy plane. For example, you might be given C using parametric equations $x = x(t), y = y(t), a \leq t \leq b$. Furthermore, Serra has told you what the height of the installation is at any point of C. This might be given to you by a function $f(x, y)$, defined on C. How much sheet metal (that is, what is the surface area) is needed for the sculpture?

One the one hand, if C were a line segment; say a line segment in the x-axis, then we could just integrate f over that line segment in the x-axis, since that integral is the area under the height function f, which gives the surface area of the installation. On the other hand, if $f(x, y)$ were constant, then the surface area of the metal would be the length of C times the height $f(x, y)$. The length of C is the arc length of C, which we know how to calculate: the arc length is equal to the integral

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

The situation we're dealing with is a mix of these two simpler cases: we are allowing the function $f(x, y)$ to vary, while letting C be a curve in the xy plane. In any case, this seems to be the type of problem integration is suited for.

The key idea behind integration is that it is the limit of approximations made by assuming f is constant over various pieces of the region we are integrating over. Suppose we make an approximation to the surface area of the metal as follows. We select various points of C, say $P_0, P_1, \ldots, P_n$, where P_0, P_n are the endpoints of C. We also require that as i increases, P_i keeps moving in the same direction. We will approximate the area of sheet metal between points P_{i-1} and P_i as follows: we pretend $f(x, y)$ is constant on this piece of C, say, equal to $f(x_i^*, y_i^*)$, for some (x_i^*, y_i^*) between P_{i-1}, P_i, and we also pretend that C a straight line from P_{i-1} to P_i. With both of these approximations in mind, the surface area of the metal from P_{i-1} to P_i is given by

$$f(x_i^*, y_i^*)\sqrt{\Delta x_i^2 + \Delta y_i^2}$$

where Δx_i is the change in the x coordinate from P_{i-1} to P_i. Now, each of these points P_i is equal to $(x(t_i), y(t_i))$, where we assume $x(t), y(t), C$ are given in such a way to ensure that t_i are an increasing sequence of numbers, with $t_0 = a, t_n = b$. Therefore, if we sum all these approximations, we end up with a Riemann sum

$$\sum_i f(x(t_i^*), y(t_i^*))\sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}\Delta t.$$

The limit of this Riemann sum is defined to be the *line integral* of f on C, and is written

$$\int_C f(x,y)\,ds.$$

On the other hand, this Riemann sum looks like a Riemann sum for the variable t, and the limit as $\Delta t \to 0$ is equal to

$$\int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

If we write C using vector notation, and let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, then this equation can be rewritten as

$$\int_a^b f(x(t), y(t))|\mathbf{r}'(t)|\,dt = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt.$$

Strictly speaking, to make sure that the line integral of $f(x,y)$ over C is equal to the expression we've given above, we want C to be smooth, which means that $\mathbf{r}'(t)$ exists and is everywhere nonzero, and $f(x,y)$ to be continuous.

Examples.

- Line integrals are genuine generalizations of definite integrals of a single variable. Suppose the curve C is an interval $[a,b]$ on the x-axis, and we have a function $f(x,0) = f(x)$ defined on that interval. Then C can be parameterized by $x(t) = t, y(t) = 0$, where $a \le t \le b$. Then the line integral of f over C is equal to

$$\int_C f(x,y)\,ds = \int_a^b f(x(t), y(t))\sqrt{1^2 + 0^2}\,dt = \int_a^b f(t,0)\,dt = \int_a^b f(t)\,dt.$$

- Arc length integrals are special cases of line integrals. Suppose we want to evaluate the line integral of the constant function $f(x,y) = 1$ on C, parameterized by $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. Then we have

$$\int_C f(x,y)\,ds = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt,$$

  which is just the formula for the arc length of C.
- One consequence of the presence of the factor of $|\mathbf{r}'(t)|$ is that line integrals are independent of the parameterization of C, at least if we restrict ourselves to parameterizations which traverse each point of C exactly once. This is good to know, since our notation for line integrals depends only on $f(x,y)$ and C, and not the choice of parameterization for C. However, when actually calculating a line integral, you will need to determine some parameterization for C.

Example. Richard Serra has given you your first assignment! He wants to install a piece of sheet metal over the curve given by the part of $y = x^2$ between $x = 0, x = 2$, and the height of this metal at (x, x^2) is given by the function $f(x,y) = f(x,x^2) = 2x$. What is the surface area of the metal you must cut out for this installation?

When starting out with a line integral problem, the very first place to start is to determine a parameterization for C, if one is not given to you. In this case, we need to parameterize the part of $y = x^2$ which lies between $x = 0$ and $x = 2$. Perhaps the most obvious parameterization for this curve is $x(t) = t, y(t) = t^2, 0 \leq t \leq 2$. Then the corresponding line integral is

$$\int_C f(x,y)\,ds = \int_0^2 2t\sqrt{1^2 + (2t)^2}\,dt.$$

This can be evaluated using a u-substitution; let $u = 4t^2 + 1, du = 8t\,dt$, so the above integral is equal to

$$\frac{1}{4}\int_1^{17} \sqrt{u}\,du = \frac{1}{4}\frac{2u^{3/2}}{3}\Big|_1^{17} = \frac{17^{3/2} - 1}{6}.$$

We've seen one interpretation of the value of a line integral: as the area of the surface over the curve C with height $f(x,y)$. Another interpretation involves mass or charge. Suppose we have a thin wire (thin enough to be thought of as a one-dimensional object) in the shape of a curve C, with density $\rho(x,y)$ at a point (x,y). Then the mass of the wire is given by the line integral

$$m = \int_C \rho(x,y)\,ds.$$

Of course, the coordinates of the center of mass are given by

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds, \overline{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds,$$

with analogous formulas for moments of inertia, etc.

Example. Suppose a wire is in the shape of the semicircle given by $x^2 + y^2 = 1, y \geq 0$, and has uniform density. What is the center of mass of the wire?

The x-coordinate is clearly 0 by symmetry. The y coordinate is something we have to calculate, though. Assume $\rho(x,y) = 1$. Then the mass of the wire is given by the arc length of C, which is evidently equal to π. We also need to calculate

$$\int_C y\,ds.$$

We can parameterize C by using $x(t) = \cos t, y(t) = \sin t, 0 \leq t \leq \pi$. Then this line integral is equal to

$$\int_0^\pi y(t)\sqrt{(-\sin t)^2 + (\cos t)^2}\,dt = \int_0^\pi \sin t\,dt = 2.$$

Therefore, the y coordinate of the center of mass is equal to $2/\pi$.

Examples.

- Let C be the line segment connecting $(1,1)$ and $(3,5)$, and let $f(x,y) = xy$. Evaluate $\int_C f(x,y)\,ds$.

  We start by finding a parameterization of C. There are a variety of ways to do this. For example, the line segment between two points P_0, P_1 always has a parameterization

  $$\ell(t) = P_0(1-t) + P_1 t, 0 \leq t \leq 1.$$

where we treat P_0, P_1 as vectors in this equation. In this problem, this gives a parameterization of

$$\ell(t) = \langle 1,1\rangle(1-t) + \langle 3,5\rangle t = \langle 1+2t, 1+4t\rangle, 0 \le t \le 1.$$

Therefore, the line integral of $f(x,y)$ over C is

$$\int_C f(x,y)\,ds = \int_0^1 (1+2t)(1+4t)\sqrt{2^2+4^2}\,dt = \int_0^1 \sqrt{20}(8t^2+6t+1)\,dt = \sqrt{20}\left(\frac{8}{3}+3+1\right) = \frac{20\sqrt{20}}{3}.$$

Of course, you can choose a different parameterization for C. For example, if you determine that C lies on the line $y-1 = 2(x-1)$, or $y = 2x-1$, then you might select the parameterization $x(t) = t, y(t) = 2t-1, 1 \le t \le 3$. Then the corresponding calculation of the line integral of f over C is

$$\int_C f(x,y)\,ds = \int_1^3 t(2t-1)\sqrt{1^2+2^2}\,dt = \int_1^3 \sqrt{5}(2t^2-t)\,dt$$
$$= \sqrt{5}\left(\frac{2t^3}{3} - \frac{t^2}{2}\bigg|_{t=1}^{t=3}\right) = \sqrt{5}((18-9/2)-(2/3-1/2)) = \frac{40\sqrt{5}}{3},$$

which is exactly the same answer we computed earlier. This illustrates the principle that it does not matter which parameterization of C you choose, so long as you select a parameterization which traverses each point of C exactly once.

- We can also calculate line integrals over curves C which lie in $\mathbb{R}^3$ (or, for that matter, in any $\mathbb{R}^n$). The formula

$$\int_C f\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$$

still holds true for any parameterization $\mathbf{r}(t), a \le t \le b$ of C. In this situation, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ will have three components, so $|\mathbf{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$. For example, let C be given by the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle, 0 \le t \le 2\pi$, so that C is one coil of a helix. Suppose that a thin wire has the shape of C, and has density $\rho(x,y,z) = z$. What is the mass of this wire?

The mass is given by the line integral

$$\int_C \rho\,ds = \int_0^{2\pi} z(t)\sqrt{x'(t)^2+y'(t)^2+z'(t)^2}\,dt.$$

Since $x'(t)^2 + y'(t)^2 + z'(t)^2 = (-\sin t)^2 + (\cos t)^2 + 1^2 = 2$, and $z(t) = t$, this integral is

$$\int_0^{2\pi} t\sqrt{2}\,dt = \frac{t^2\sqrt{2}}{2}\bigg|_0^{2\pi} = 2\sqrt{2}\pi^2.$$