Surface Integrals

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Scalar Surface Integral

The **scalar surface integral** of a continuous function f along a smooth parametrized surface $\mathbf{X}(s,t)$ over a bounded region D is

$$\begin{aligned}\iint_{\mathbf{X}} f\,dS &= \iint_D f(\mathbf{X}(s,t))\|\mathbf{T}_s \times \mathbf{T}_t\|\,dsdt \\ &= \iint_D f(\mathbf{X}(s,t))\|\mathbf{N}(\mathbf{s},\mathbf{t})\|\,dsdt\end{aligned}$$

Vector Surface Integral

The **vector surface integral** of a continuous vector field $\mathbf{F}(x, y, z)$ along a smooth parametrized surface $\mathbf{X}(s, t)$ over a bounded region D is

$$\boxed{\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S} = \iint_{D} \mathbf{F}(\mathbf{X}(s, t)) \cdot \mathbf{N}(s, t)\, ds dt}$$

where $\mathbf{N}(s, t) = \mathbf{T}_s \times \mathbf{T}_t$.

Orientation of a surface

A **smooth orientable surface** is a two-sided surface S that has a tangent plane at every point (x, y, z) on S (except boundary points) and at each point there are two normal vectors $\mathbf{N}$ and $-\mathbf{N}$. In other words the surface has two sides. The choice of $\mathbf{N}$ gives S an orientation.

REMARKS: (1) The Möbious strip is an example of an **nonorientable** surface, it has only one side. We can only define the surface integrals for orientable surfaces!

http://mathworld.wolfram.com/MoebiusStrip.html

(2) A **closed surface** is a surface that encloses a volume. In this case one takes the convention that the **positive orientation** is the one where the normals point away from the volume (outward) and the **negative orientation** is the one where the normals point towards the inside of the volume (inward).

Scalar Surface Integrals are independent of parametrization

Theorem: Let $\mathbf{X} : D_1 \to \mathbf{R}^3$ be a smooth parametrized surface and f any continuous function with domain containing $\mathbf{X}(D_1)$. If $\mathbf{Y}$ is a smooth reparametrization of X, then

$$\iint_{\mathbf{Y}} f\, dS = \iint_{\mathbf{X}} f\, dS.$$

Vector Surface Integrals and Reparametrizations

Let Y be a reparametrization of the smooth orientable surface X. Then

- If $\mathbf{Y}$ is orientation preserving, then
$$\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$

- If $\mathbf{Y}$ is orientation reversing, then
$$\iint_{\mathbf{Y}} \mathbf{F} \cdot d\mathbf{S} = -\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$$