Scalar and Vector Line Integrals

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Scalar Line Integrals

Let $\mathbf{x} : [a, b] \longrightarrow \mathbb{R}^n$ be a C^1 path. And $f : X \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function such that the domain X contains the image of $\mathbf{x}$. The **scalar line integral** of f along $\mathbf{x}$ is

$$\boxed{\int_a^b f(\mathbf{x}(t))\|\mathbf{x}'(t)\|dt}$$

NOTATION: This integral is usually written $\int_{\mathbf{x}} f\, ds$.

Vector Line Integral

The **vector line integral** of $\mathbf{F}$ along $\mathbf{x} : [a, b] \longrightarrow \mathbb{R}^n$ is

$$\boxed{\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\, dt}$$

Physical Interpretation of Line Integrals

If $\mathbf{F}$ is a force field in space, then the vector line integral is the **work** done by $\mathbf{F}$ on a particle as the particle moves along the path $\mathbf{x}$.

Tangent and Vector Line Integral

Let $\mathbf{x} : [a, b] \longrightarrow \mathbb{R}^n$ with $\mathbf{x}'(t) \neq 0$ in $[a, b]$.

Unit tangent vector T:

$$\mathbf{T}(t) = \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|}.$$

Then

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} (\mathbf{F} \cdot \mathbf{T})\, ds$$

Reparametrizations of Paths

We say that $\mathbf{y} : [c, d] \to \mathbb{R}^n$ is a **reparametrization** of the path $\mathbf{x} : [a, b] \to \mathbf{R}^n$ if both describe the same curve.

Mathematically, this means that there is a one-to-one and onto C^1 function $u : [a, b] \to [c, d]$ such that $\mathbf{x}(u(t)) = y(t)$ and the inverse of u is also C^1.

Scalar Line Integrals
Do NOT depend on parametrization

Theorem: If $\mathbf{y}$ is a reparametrization of $\mathbf{x}$ then

$$\int_{\mathbf{y}} f\,ds = \int_{\mathbf{x}} f\,ds$$

Vector Line Integrals and parametrizations

If $\mathbf{y}$ is a reparametrization of $\mathbf{x}$. Then

- If $\mathbf{y}$ is orientation-preserving, then $\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$

- If $\mathbf{y}$ is orientation-reversing, then $\int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s} = -\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$