Triple Integrals

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Closed Box

We are interested in **closed boxes** B in $\mathbb{R}^3$ such that its faces are parallel to the coordinate planes:

$$B = \{(x, y, z) \,|\, a \leq x \leq b, c \leq y \leq d, p \leq z \leq q\}.$$

or

$$B = [a, b] \times [c, d] \times [p, q].$$

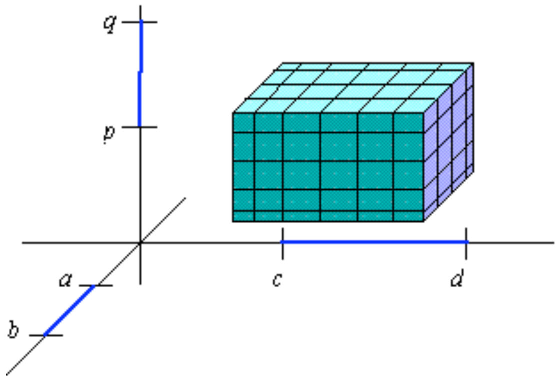
Partition of a closed box

A **partition** of a closed box B of order n break up B into n^3 subboxes. We divide each of the intervals that make up B into n subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$
$$c = y_0 < y_1 < y_2 < \cdots < y_n = d$$
$$p = z_0 < z_1 < z_2 < \cdots < z_n = q$$

We also have $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$ and $\Delta z_k = z_k - z_{k-1}$

Partition of a box



Riemann Sum

Let $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ be a subbox of B and let $\mathbf{c}_{ijk}$ be a point in this box then the **Riemann sum** is

$$S = \sum_{i,j,k}^{n} f(\mathbf{c}_{ijk}) \Delta x_i \Delta y_j \Delta z_k.$$

Triple Integral

The **triple integral** over a box is defined as the limit of the Riemann sum

$$\iiint_B f(x,y,z)\,dV = \iiint_B f(x,y,z)\,dxdydz$$

$$= \lim_{\Delta x_i, \Delta y_j, \Delta z_k \to 0} \sum_{i,j,k=1}^{n} f(\mathbf{c_{ijk}})\Delta x_i \Delta y_j \Delta z_k$$

Integrability of functions

If f is bounded on B and the set of discontinuities of f on B has **zero volume**, then $\iiint_B f(x,y,z)\,dV$ exits.

Fubini's Theorem

Let f be integrable over $B = [a, b] \times [c, d] \times [p, q]$, then

$$\iiint_B f \, dV$$

$$= \int_a^b \int_c^d \int_p^q f \, dz dy dx = \int_a^b \int_p^q \int_c^d f \, dy dz dx$$
$$= \int_c^d \int_a^b \int_p^q f \, dz dx dy = \int_c^d \int_p^q \int_a^b f \, dx dz dy$$
$$= \int_p^q \int_a^b \int_c^d f \, dy dx dz = \int_p^q \int_c^d \int_a^b f \, dx dy dz$$

Elementary Regions

Type 1:

(a) W contains points such that

$$\phi(x,y) \leq z \leq \psi(x,y)$$
$$\gamma(x) \leq y \leq \delta(x)$$
$$a \leq x \leq b$$

(b) W contains points such that

$$\phi(x,y) \leq z \leq \psi(x,y)$$
$$\alpha(y) \leq x \leq \beta(y)$$
$$c \leq y \leq d$$

Elementary Regions

Type 2:

(a) W contains points such that

$$\alpha(y,z) \le x \le \beta(y,z)$$
$$\gamma(z) \le y \le \delta(z)$$
$$p \le z \le q$$

(b) W contains points such that

$$\alpha(y,z) \le x \le \beta(y,z)$$
$$\phi(y) \le z \le \psi(y)$$
$$c \le y \le d$$

Elementary Regions

Type 3:

(a) W contains points such that

$$\gamma(x,z) \leq y \leq \delta(x,z)$$
$$\alpha(z) \leq x \leq \beta(z)$$
$$p \leq z \leq q$$

(b) W contains points such that

$$\gamma(x,z) \leq y \leq \delta(x,z)$$
$$\phi(x) \leq z \leq \psi(x)$$
$$a \leq x \leq b$$

NOTE: **Type 4** W is of all three types described before.

Triple integrals over elementary regions

If W is of type 1(a) then

$$\iiint_W f\,dV = \int_a^b \int_{\gamma(x)}^{\delta(x)} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\,dzdydx$$

If W is of type 1(b) then

$$\iiint_W f\,dV = \int_c^d \int_{\alpha(y)}^{\beta(y)} \int_{\phi(x,y)}^{\psi(x,y)} f(x,y,z)\,dzdxdy$$

Triple integrals over elementary regions

If W is of type 2(a) then

$$\iiint_W f\,dV = \int_p^q \int_{\gamma(z)}^{\delta(z)} \int_{\alpha(y,z)}^{\beta(y,z)} f(x,y,z)\,dxdydz$$

If W is of type 2(b) then

$$\iiint_W f\,dV = \int_c^d \int_{\phi(y)}^{\psi(y)} \int_{\alpha(y,z)}^{\beta(y,z)} f(x,y,z)\,dxdzdy$$

Triple integrals over elementary regions

If W is of type 3(a) then

$$\iiint_W f\,dV = \int_p^q \int_{\alpha(z)}^{\beta(z)} \int_{\gamma(x,z)}^{\delta(x,z)} f(x,y,z)\,dydxdz$$

If W is of type 3(b) then

$$\iiint_W f\,dV = \int_a^b \int_{\phi(y)}^{\psi(y)} \int_{\gamma(x,z)}^{\delta(x,z)} f(x,y,z)\,dydzdx$$