Solutions to Practice Exam 2

Problem 1: For each of the following, set up (but do not evaluate) iterated integrals or quotients of iterated integral to give the indicated quantities:

Problem 1a: The average temperature at points in the region W lying between the paraboloids $z = 12 - x^2 - y^2$ and $z = 2(x^2 + y^2)$ if the temperature is inverse proportional to the distance from the z-axis. (Use cylindrical coordinates.)

Solution: We first find where the two paraboloids intersect. We want to solve $12 - x^2 - y^2 = 2(x^2 + y^2)$, which is equivalent to $12 = 3x^2 + 3y^2$ or $x^2 + y^2 = 4$. Thus, the two paraboloids intersect above the circle $x^2 + y^2 = 4$ (the circle centered at the origin with radius 2) with value $z = 8$. Since $z = 12 - x^2 - y^2$ is a paraboloid opening downward and $z = 2(x^2 + y^2)$ is a paraboloid opening upward, it follows that the projection of W in the xy-plane is the disk bounded by $x^2 + y^2 = 4$. In cylindrical coordinates, the paraboloid $z = 12 - x^2 - y^2$ is given by $z = 12 - r^2$ and the paraboloid $z = 2(x^2 + y^2)$ is given by $z = 2r^2$. Furthermore, out temperature function is given by $T(x, y) = \frac{k}{\sqrt{x^2+y^2}} = \frac{k}{r}$, so the integral of T over W equals

$$\int\int\int_W T(x,y,z)\ dV = \int_0^{2\pi}\int_0^2\int_{2r^2}^{12-r^2} \frac{k}{r}\cdot r\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^2\int_{2r^2}^{12-r^2} k\ dz\ dr\ d\theta$$

The volume of W is given by

$$\int\int\int_W 1\ dV = \int_0^{2\pi}\int_0^2\int_{2r^2}^{12-r^2} r\ dz\ dr\ d\theta$$

Thus, the average temperature equals

$$\frac{\int_0^{2\pi}\int_0^2\int_{2r^2}^{12-r^2} k\ dz\ dr\ d\theta}{\int_0^{2\pi}\int_0^2\int_{2r^2}^{12-r^2} r\ dz\ dr\ d\theta}$$

Problem 1b: The moment of inertia about the z-axis of the region lying outside the double cone $z^2 = x^2 + y^2$ and inside the ball $x^2 + y^2 + z^2 = 1$ if $\delta(x, y, z) = k$.

Solution: We use spherical coordinates. Notice that the ball has spherical coordinates $\rho = 1$. Since $x^2 + y^2 = \rho^2 \sin^2\varphi$ in spherical coordinates, notice that $z^2 = x^2 + y^2$ in spherical coordinates is

$$\rho^2\cos^2\varphi = \rho^2\sin^2\varphi$$

which is equivalent to

$$\sin^2\varphi = \cos^2\varphi$$

which in turn is equivalent to

$$\tan^2\varphi = 1$$

or more simply

$$\tan\varphi = \pm 1$$

Since $0 \le \varphi \le \pi$, the double cone is simply given by $\varphi = \frac{\pi}{4}$ or $\varphi = \frac{3\pi}{4}$. Therefore, the integral giving the moment of inertia about the z-axis is

$$\begin{aligned}\int\int\int_R (x^2+y^2)\delta(x,y,z)\ dV &= \int\int\int_R k(x^2+y^2)\ dV \\ &= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^1 k(\rho^2\sin^2\varphi)(\rho^2\sin\varphi)\ d\rho\ d\varphi\ d\theta \\ &= \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^1 k\rho^4\sin^3\varphi\ d\rho\ d\varphi\ d\theta\end{aligned}$$

Problem 2: The base of a fence lies along the part of the parabola $y = x^2$, $1 \le x \le 2$, and the height of the fence at the point (x, y) is given by $h(x, y) = x$ (in feet). Find the area of the fence.

Solution: We parametrize the curve by $\mathbf{x}(t) = (t, t^2)$ for $1 \le t \le 2$. We then have $\mathbf{x}'(t) = (1, 2t)$ for all t, hence

$$||\mathbf{x}'(t)|| = \sqrt{1^2 + (2t)^2} = \sqrt{1+4t^2}$$

for all t. We also have

$$h(\mathbf{x}(t)) = h(t, t^2) = t$$

for all t. Using the substitution $u = 1 + 4t^2$, we have $du = 8t\ dt$ and hence

$$\begin{aligned}\int_{\mathbf{x}} h\ ds &= \int_1^2 h(\mathbf{x}(t))||\mathbf{x}'(t)||\ dt \\ &= \int_1^2 t\sqrt{1+4t^2}\ dt \\ &= \int_5^{17} \frac{\sqrt{u}}{8}\ du \\ &= (\frac{2}{3}\cdot\frac{u^{3/2}}{8})|_{u=5}^{u=17} \\ &= \frac{u^{3/2}}{12}|_5^{17} \\ &= \frac{17^{3/2}}{12} - \frac{5^{3/2}}{12} \\ &= \frac{17^{3/2} - 5^{3/2}}{12}\end{aligned}$$

Problem 3: Consider the transformation $T(u, v) = (u^2\cos(v), u^2\sin(v))$. Describe (e.g., by a picture) how T transforms the rectangle $[0, 2] \times [0, \frac{\pi}{2}]$.

Solution: See the images file for a sketch.

- $T(u, 0) = (u^2 \cdot 1, u^2 \cdot 0) = (u^2, 0)$ for $0 \le u \le 2$. Thus, T maps the bottom of the rectangle onto $[0, 4]$ of the x-axis.
- $T(2, v) = (2^2\cos v, 2^2\sin v) = (4\cos v, 4\sin v)$ for $0 \le v \le \frac{\pi}{4}$. Thus, T maps the right side of the rectangle onto the circle of radius 4 in the first quadrant.

- $T(u, \frac{\pi}{2}) = (u^2 \cdot 0, u^2 \cdot 1) = (0, u^2)$ for $0 \le u \le 2$. Thus, T maps the top of the rectangle onto $[0, 4]$ of the y-axis.

- $T(0, v) = (0^2 \cos v, 0^2 \sin v) = (0, 0)$ for $0 \le v \le \frac{\pi}{4}$. Thus, T collapses the left side of the rectangle onto the origin.

In general, for any $(u, v) \in [0, 2] \times [0, \frac{\pi}{2}]$, the point $T(u, v)$ is on the circle of radius u^2 centered at the origin because

$$\sqrt{(u^2 \cos v)^2 + (u^2 \sin v)^2} = \sqrt{u^4 \cos^2 v + u^4 \sin^2 v} = \sqrt{u^4} = u^2$$

and $T(u, v)$ is makes an angle of v with the x-axis because

$$\frac{u^2 \sin v}{u^2 \cos v} = \tan v$$

Summing up, the point $T(u, v)$ has polar coordinates (u^2, v). Thus, T maps $[0, 2] \times [0, \frac{\pi}{2}]$ onto the part of the disk enclosed by $x^2 + y^2 = 4$ in the first quadrant.

Problem 4: A snowball rolls downhill along a curvy path given by $\mathbf{x}(t) = (t, t^3, 25 - t^2)$, $0 \le t \le 5$ (where the first two coordinates denote it's position east-west and north-south and the third coordinate denotes its elevation). The snowball steadily increases in size as it rolls so that its weight at time t is given by $2 + 3t$ pounds. Find the total work done by gravity on the snowball. (The gravitational force has magnitude equal to the weight of the snowball and points directly down, i.e., in the direction of $-\mathbf{k}$.)

Solution: Notice that for every t, we have

$$\mathbf{F}(\mathbf{x}(t)) = -(2 + 3t)\mathbf{k}$$

and also

$$\mathbf{x}'(t) = (1, 3t^2, -2t)$$

and hence

$$\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) = -(2 + 3t)(-2t) = 4t + 6t^2$$

Therefore, the total done by gravity on the snowball equals

$$\begin{aligned} \int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} &= \int_0^5 (4t + 6t^2)\ dt \\ &= (2t^2 + 2t^3)|_0^5 \\ &= 50 + 250 \\ &= 300 \end{aligned}$$

Problem 5: Let D be a region bounded by a simple closed curve C in the xy-plane. Use Green's theorem to prove that the coordinates of the centroid of D are given by

$$\bar{x} = \frac{1}{2A} \oint_C x^2\, dy$$

and

$$\bar{y} = \frac{1}{2A} \oint_C -y^2\, dx$$

where A is the area of D.

Solution: Suppose that $\delta(x, y) = k$. We have

$$\begin{aligned}
\frac{1}{2A}\oint_C x^2\,dy &= \frac{1}{2A}\int\int_D (\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(0))\;dA && \text{(by Green's Theorem)}\\
&= \frac{1}{2A}\int\int_D (2x - 0)\;dA\\
&= \frac{2}{2A}\int\int_D x\;dA\\
&= \frac{\int\int_D x\;dA}{A}\\
&= \frac{k\int\int_D x\;dA}{k\int\int_D 1\;dA}\\
&= \frac{\int\int_D kx\;dA}{\int\int_D k\;dA}\\
&= \bar{x}
\end{aligned}$$

and also

$$\begin{aligned}
\frac{1}{2A}\oint_C -y^2\,dx &= \frac{1}{2A}\int\int_D (\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(-y^2))\;dA && \text{(by Green's Theorem)}\\
&= \frac{1}{2A}\int\int_D (0 - (-2y))\;dA\\
&= \frac{2}{2A}\int\int_D y\;dA\\
&= \frac{\int\int_D y\;dA}{A}\\
&= \frac{k\int\int_D y\;dA}{k\int\int_D 1\;dA}\\
&= \frac{\int\int_D ky\;dA}{\int\int_D k\;dA}\\
&= \bar{y}
\end{aligned}$$

Problem 6: Each of the following is difficult or impossible to evaluate directly but can be computed by other methods. Compute each one and justify your method.

Problem 6a: $\int_C 2x\,dx + cos(y^2)\,dy$ where C is the semicircle $x^2+y^2 = 1$, $y \geq 0$, traced from $(-1, 0)$ to $(1, 0)$.

Solution: Notice that

$$\frac{\partial}{\partial x}(\cos(y^2)) = 0 = \frac{\partial}{\partial y}(2x)$$

so since we are working on the simply-connected region $\mathbb{R}^2$, it follows that the vector field $\mathbf{F}(x, y) = (2x, \cos(y^2))$ is conservative. A potential is quite hard to come by because we can not integrate $\cos(y^2)$ with respect to y. However, we know that $\mathbf{F}$ has path-independent line integrals, so we may evaluate the line integral by choosing the straight line path from $(-1,0)$ to $(1,0)$. We parametrize this path by $\mathbf{x}(t) = (t,0)$ for $-1 \leq t \leq 1$. Since $\mathbf{x}'(t) = (1,0)$ for every t, we have

$$\begin{aligned}\int_C 2x\,dx \,+\, cos(y^2)\,dy &= \int_{\mathbf{x}} 2x\,dx \,+\, cos(y^2)\,dy \\ &= \int_{-1}^{1} (2t \cdot 1 + \cos(0^2) \cdot 0)\ dt \\ &= \int_{-1}^{1} 2t\ dt \\ &= t^2|_{-1}^{1}\ dt \\ &= 1 - 1 \\ &= 0\end{aligned}$$

Problem 6b: $\int_C 3x^2y^2\,dx + 2x^3y\,dy$ where C is the path $\mathbf{x}(t) = (e^{t^2}, t^2)$, $0 \leq t \leq 1$

Solution: Notice that

$$\frac{\partial}{\partial x}(2x^3y) = 6x^2y = \frac{\partial}{\partial y}(3x^2y^2)$$

so since we are working on the simply-connected region $\mathbb{R}^2$, it follows that the vector field $\mathbf{F}(x,y) = (3x^2y^2, 2x^3y)$ is conservative. We look for a potential function f. We need our f to satisfy

1. $f_x(x,y) = 3x^2y^2$
2. $f_y(x,y) = 2x^3y$

Integrating (1) with respect to x, we know that

$$f(x,y) = x^3y^2 + g(y)$$

for some function g. Now taking the partial of this with respect to y, we see that

$$f_y(x,y) = 2x^3y + g_y(y)$$

and comparing that with (2) above we can conclude that $g_y(y) = 0$ for all y. Therefore, $g(y) = c$ for some constant c, and hence for every c

$$f(x,y) = x^3y^2 + c$$

is a potential function for $\mathbf{F}$. We work with the potential function $f(x,y) = x^3y^2$. Notice that the initial point of our path is $\mathbf{x}(0) = (e^0, 0^2) = (1,0)$ and the terminal point of our path is $\mathbf{x}(1) = (e^1, 1^2) = (e,1)$. Therefore, since $\mathbf{F}$ is conservative with potential function $f(x,y) = x^3y^2$, it follows that

$$\begin{aligned}\int_C 3x^2y^2\,dx + 2x^3y\,dy &= f(e,1) - f(1,o) \\ &= e^3 \cdot 1^2 - 1^3 \cdot 0^2 \\ &= e^3\end{aligned}$$