Differential form:

$\sigma(t) = (x(t), y(t), z(t)) : [a,b] \to \mathbb{R}^3$ a path

$F(x,y,z) = (M(x,y,z), N(x,y,z), P(x,y,z))$ a vector field

Then

$$\int_\sigma F \cdot ds = \int_a^b \left(M(x,y,z)\cdot x'(t) + N(x,y,z)\cdot y'(t) + P(x,y,z) z'(t)\right) dt$$

Notice
$x'(t)\,dt = dx$
$y'(t)\,dt = dy$
$z'(t)\,dt = dz$
so

$$= \int_a^b M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$$

e.g. We compute $\int_\sigma (y+z)dx + (x+z)dy + (x+y)dz$

where $\sigma(t) = (t, t^2, t^3)$, $0 \le t \le 1$

Soln: $\int_0^1 (t^2+t^3)dt + (t+t^3)2t\,dt + (t+t^2)3t^2\,dt = 3.$

What happens if you reparameterize?

$\sigma(t) : [0, 2\pi] \to \mathbb{R}^2$ $(\cos t, \sin t)$ — a circle traced out counterclockwise at 1 radian/sec

vs $\psi(t) : [0, \pi] \to \mathbb{R}^2$ $(\cos 2t, \sin 2t)$ — a circle traced out counterclockwise at 2 radian/sec

vs $\mu(t) = [0, 2\pi] \to \mathbb{R}^2$ $(-\cos t, -\sin t)$ — a circle traced out clockwise at 1 radian/sec

Def. Let $\sigma : [a,b] \to \mathbb{R}^n$ be a path. We say that $\varphi : [c,d] \to \mathbb{R}^n$ is a reparameterization of σ is there is some $u : [c,d] \to [a,b]$ s.t. ~~$\sigma(t)$~~ ~~$\varphi(u(t))$~~ =

$$\underbrace{\sigma(\underbrace{u(\underset{[c,d]}{t})}_{[a,b]})}_{\mathbb{R}^n} = \varphi(t)$$

and so that $u^{-1} : [a,b] \to [c,d]$.

e.g. $\sigma(t) = (1+2t, 2-t, 3+5t)$ $0 \le t \le 1$

~~One of~~ The following $\varphi(t) = (1+2t^2, 2-t^2, 3+5t^2)$ $0 \le t \le 1$

$\psi(t) = (3-2t, 1+t, 8-5t)$ $0 \le t \le 1$ are reparameterizations of $\sigma(t)$.

Method I of verifying this: draw $\sigma(t)$, $\varphi(t)$ and $\psi(t)$ and see that they produce the same curve without repeating points.

Method II of verifying this: realize that $\varphi(t) = \sigma(t^2)$ and $\psi(t) = \sigma(1-t)$.

Orientations of paths:

$\sigma(t)$ and $\varphi(t)$ ~~have~~ are drawn going in the same direction, i.e., $\sigma(0) = \varphi(0)$ and $\sigma(1) = \varphi(1)$, i.e., they have the same starting point and the same endpoint.

On the other hand $\sigma(t)$ and $\psi(t)$ are drawing going in the opposite directs; i.e., $\sigma(0) = \psi(1)$ and $\sigma(1) = \psi(0)$; ~~the~~ i.e. they swap endpoints and starting points.

Def'n: ~~Let φ~~ $\varphi(t) = \sigma(u(t))$ is a reparameterization, ~~then~~ and

$u: [c,d] \to [a,b]$.

(a) If $u(c)=a$ and $u(d)=b$, we say u (or φ) is <u>orientation preserving</u>.

(b) if $u(c)=b$ and $u(d)=a$, we say u (or φ) is <u>orientation reversing</u>.

eg $\sigma: [a,b] \to \mathbb{R}^n$ a path. Define $\sigma_{opp}: [a,b] \to \mathbb{R}^n$ by

$$\sigma_{opp}(t) = \sigma(a+b-t).$$

What does this do? ~~At $t=a$ σ_{opp}~~

$$\sigma_{opp}(a) = b$$
$$\sigma_{opp}(b) = a$$

So, $\sigma_{opp}(t)$ is the reparameterization of $\sigma(t)$ given by drawing it from $\sigma(b)$ to $\sigma(a)$ rather than from $\sigma(a)$ to $\sigma(b)$

Note: A reparameterization can change the orientation but also the speed a which ~~it~~ the curve is traced out.

Nevertheless,

Thm $\sigma: [a,b] \to \mathbb{R}^n$, $\varphi: [c,d] \to \mathbb{R}^n$, a reparameterization of σ. Let $f: \mathbb{R}^n \to \mathbb{R}$ be continuous.

Then

$$\int_\sigma f\, ds = \int_\varphi f\, ds.$$

I.e., the reparameterization doesn't change the (scalar) line integral.

~~Pf~~

Pf sketch: Let $\varphi(t) = \sigma(u(t))$. Then

speed of φ $= \| \varphi'(t) \| = \| \sigma'(u(t))\, u'(t) \|$

$= \|\sigma'(u(t))\| \, |u'(t)|$

$=$ speed of $\sigma \cdot |u'(t)|$

Now

$$\int_\varphi f \, ds = \int_c^d f(\varphi(t)) \cdot \overbrace{\cancel{\text{speed of } \varphi}}^{\|\varphi'(t)\|} \, dt$$

$$= \int_c^d f(\varphi(t)) \, |u'(t)| \, \|\sigma'(u(t))\| \, dt$$

If φ is orientation reversing, then $u(c) = b$, $u(d) = a$, $|u'(t)| = -u'(t)$

~~see~~ since $u'(t) \leq 0$. So using ~~chain~~ u-substitution

$$\int_c^d f(\cancel{\varphi(t)}\sigma(u(t))) \, |u'(t)| \, \|\sigma'(u(t))\| \, dt$$

$$= \int_b^a f(\sigma(u))(-du) \, \|\sigma'(u)\|$$

$$= \int_a^b f(\sigma(u)) \, \|\sigma'(u)\| \, du = \int_\sigma f \, ds.$$

$\boxed{}$

Thm: Let $\sigma : [a,b] \to \mathbb{R}^n$ a pc $\mathcal{R}$. $\varphi : [c,d] \to \mathbb{R}^n$ a ~~path~~ reparameterization of σ.

(1) If ~~σ~~ φ is orientation preserving $\int_\sigma F \cdot ds = \int_\varphi F \cdot ds$

(2) If φ is orientation-reversing $\int_\sigma F \cdot ds = -\int_\varphi F \cdot ds$

Pf observation: You have just $u'(t)$ rather than $|u'(t)|$.

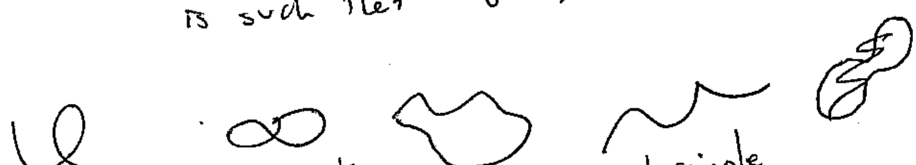
Significance of theorems

- Since $\int_\sigma f\,ds$ and $\int_\sigma \mathbf{F}\cdot d\mathbf{s}$ are (pretty-much) independent of path we can define integrals over curves, not just parameterized paths.

<u>Curves</u>: A curve is the image of a parameterized path.

A curve is <u>simple</u> if it has no intersection.

A curve is <u>closed</u> if its underlying parameterization $\sigma : [a,b] \to \mathbb{R}^n$ is such that $\sigma(a) = \sigma(b)$.



not simple, not closed — not simple, closed — simple, closed — ~~not~~ simple, not closed

Simple curves, whether closed or not, allow for a choice of orientation

<u>Scalar line integral</u>: orientation has no effect. We can define $\int_C f\,ds$ for an unoriented curve C.

<u>Vector line integral</u>: We can only define it for <u>oriented</u> curves.

<u>e.g</u> Let the curve C be the perimeter of the unit square in $\mathbb{R}^2$.
Evaluate the line integral: $\int_C x^2\,dx + xy\,dy$.

Find a parameterization for C: e.g $t \mapsto \begin{cases} (t,0) & 0 \le t \le 1 \\ (1,t-1) & 1 \le t \le 2 \\ (3-t,1) & 2 \le t \le 3 \\ (0,4-t) & 3 \le t \le 4 \end{cases}$

$$\int_C x^2dx + xydy = \int_0^1 t^2\,dt + \int_0^1 0\,d0 + \int_1^2 1\cdot d1 + \int_1^2 (t-1)\,dt + \int_2^3 (3-t)^2(-dt) + \int_2^3 (3-t)\,d1 + \int_3^4 0^2\,d0 + \int_3^4 0\cdot(4-t)(-dt) = \frac{1}{2}$$