Gauss's Theorem:

closed surface integral $\Longleftrightarrow$ triple integral

Thm: ~~(Gauss's Thm)~~: Let D be a solid region in $\mathbb{R}^3$ whose boundary consists of finitely many piecewise smooth, closed orientable surfaces, each of which is oriented by unit normal vectors that point away from D. Let F be a smooth vector field defined on D.

Then

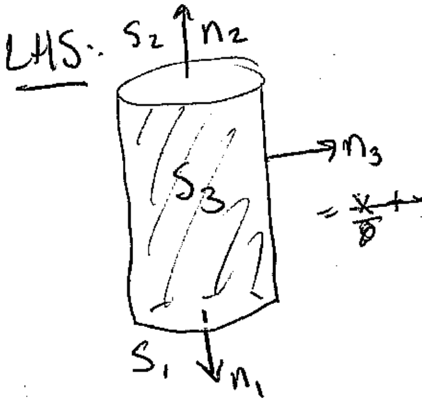
$$\oiint_{\partial D} F\cdot dS = \iiint_D \nabla\cdot F \, dV.$$

E.g. Let $F(x,y,z) = (x,y,z)$.

D be the solid cylinder of height b and radius a.

Verify Gauss's Theorem.

LHS:



$= \cancel{\frac{x+y}{a}}\ \frac{(x,y)}{a}$

$$\oiint_{\partial D} F\cdot dS = \iint_{S_1} F\cdot dS + \iint_{S_2} F\cdot dS + \iint_{S_3} F\cdot dS$$

$$= \iint_{S_1} (x,y,z)\cdot(0,0,-1)\,dS + \iint_{S_2} (x,y,z)\cdot(0,0,1)\,dS$$

$$+ \iint_{S_3} (x,y,z)\cdot\left(\frac{x}{a},\frac{y}{a},0\right) dS$$

$$= \underbrace{\iint 0 \, dS}_{\text{since } z=0 \text{ in } S_1} + \underbrace{\iint b\, dS}_{\text{since } z=b \text{ in } S_2} + \underbrace{\iint \overset{\frac{a^2}{a}}{\frac{x^2+y^2}{a}}\, dS}_{\text{since } x^2+y^2=a^2 \text{ on } S_3}$$

$= b\cdot$ area of $S_2 + a\cdot$ area of S_3

$= 3\pi a^2 b$

RHS:

$$\iiint_S \nabla\cdot F \, dV : \quad \nabla\cdot F = \frac{\partial}{\partial x}x + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

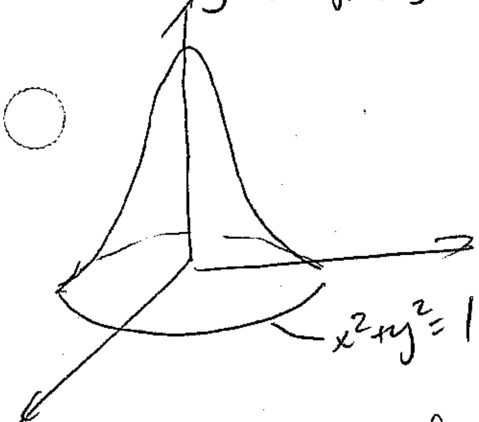
$$= \iiint_S 3\, dV = 3 \cdot \text{volume of the cylinder}$$

$$= 3 \cdot (\text{area of base})(\text{height}) = 3\pi a^2 b.$$

e.g. Cool way to use Gauss's Theorem:

$$F = e^y \cos z \,\hat{\imath} + \sqrt{x^3+1}\,\sin z \,\hat{\jmath} + (x^2+y^2+3)\hat{k}$$

S the graph of $z = (1-x^2-y^2)e^{1-x^2-3y^2}$ for $z \geq 0$



$$\iint_S F \cdot dS$$

gets pretty gross pretty quickly

OTOH, Define S' to be S and the unit disk D: $x^2+y^2 \leq 1$.

Now ~~is~~ S' ~~is a closed~~ the boundary of a solid region W in $\mathbb{R}^3$.

If we orient the unit disk with normal vector $-\hat{k}$, we can apply

Green's Theorem:

$$\cancel{\iiint_{S'}} : \iint_S F\cdot dS + \iint_D F\cdot dS = \oiint_{S'} F\cdot dS = \iiint_W \nabla\cdot F \, dV$$

$\nabla \cdot F = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0.$

So $\iint_S F \cdot dS + \iint_D F \cdot dS = 0. \Leftrightarrow \iint_S F \cdot dS = -\iint_S F \cdot dS$

Now

$$- \iint_D F \cdot dS = \cancel{\int_0^{2\pi}} \int_0^1 \quad \cancel{r\,dz}\,d\theta$$

$$= -\iint_D F \cdot (-k) \quad dS$$

$$= +\iint (x^2+y^2+3) \cancel{dS}\, dx\,dy$$

$$= +\int_0^1\int_0^{2\pi} (r^2+3)\, r\, d\theta\, dr$$

$$= 2\pi \int_0^1 r^3 + 3r\, dr = 2\pi \left(\frac{r^4}{4} + \frac{3r^2}{2} \right)\Big|_0^1 =$$

$$= 2\pi \left(\frac{1}{4} + \frac{3}{2} \right) = \frac{7}{2}\pi$$

So $\iint_S F \cdot dS = \frac{7}{2}\pi$

Meaning of div & curl:

~~div F(x,y,z)~~ Let P be a pt in $\mathbb{R}^3$, S_a the sphere of radius a centered at P, ~~then~~ oriented w/ outward normals. F a vector field then

$$\operatorname{div} F(P) = \lim_{a\to 0} \frac{3}{4\pi a^3} \iint_{S_a} F\cdot dS.$$

Interpretation: $\iint_{S_a} F\cdot dS$ is the flux across S_a

$$\frac{\iint_{S_a} F\cdot dS}{\frac{4}{3}\pi a^3}$$ is the flux per unit volume, i.e., flux density.

Pf: Using Gauss's theorem and a mean value theorem for triple integrals.

Let F be a vector field, P a point in $\mathbb{R}^3$. $\hat{n}$ some unit vector, C_a a circle ~~in the plane~~ centered at P in the plane for which $\hat{n}$ is the normal vector. Then

$$n\cdot \operatorname{curl} F(P) = \lim_{a\to 0} \frac{1}{\pi a^2} \oint_{C_a} F\cdot ds$$

where C_a has orientation induced by the right hand rule.

Pf: Follows from Stokes's Theorem and a mean value Theorem for surface integrals.

Interpretation: $\oint_{C_a} F \cdot ds$ is the circulation around C_a.

$$\frac{\oint_{C_a} F \cdot ds}{\pi a^2}$$ is the circulation per unit area.

limit is the circulation at that point P, so how much things are spinning, its tangential motion.