eg $\int_{-\infty}^{\infty} e^{-x^2}dx = \cancel{2\lim_{a\to\infty}\int_0^a e^{-x^2}dx} = \lim_{a\to\infty}\int_0^a e^{-x^2}dx + \lim_{a\to\infty}\int_{-a}^0 e^{-x^2}dx$

Let $h(x,y) = e^{-x^2}\cdot e^{-y^2} = e^{-(x^2+y^2)}$, a bell surface.

Slice along $y=0$ get the bell curve. Slice along $y=c$, get $e^{-c^2}\cdot e^{-x^2}$, a scaled bell curve.

Suppose $\int_{-\infty}^{\infty} e^{-x^2}dx = A$. ~~What is A?~~ Compute A.

Note $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy = \int_{-\infty}^{\infty} e^{-y^2}\int_{-\infty}^{\infty} e^{-x^2}dx\,dy$

$$= \int_{-\infty}^{\infty} e^{-y^2} A\,dy$$

$$= A\int_{-\infty}^{\infty} e^{-y^2}dy = A\cdot A = A^2.$$

On the other hand convert to polar coordinates: The plane in polar coordinates is $r\in[0,\infty)$ and $\theta\in[0,2\pi)$.

So

$$\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta = \int_0^{2\pi} -\frac{1}{2}e^{-r^2}\Big|_0^{\infty}\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\cdot 0 - \frac{1}{2}e^0\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\,d\theta = \pi.$$

So $A^2 = \pi$ and thus $A = \sqrt{\pi} \approx 1.7724\ldots$

The graph e^{-x^2} is the so-called bell-curve, Plinko, how things are distributed in nature.

Change of variables for triple integrals

ingredients – $T: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w))$

– Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \det DT(u,v,w)$

– Thus $\iiint_W f(x,y,z)\, dx\, dy\, dz$

$$= \iiint_{W^* = T(W)} f(x(u,v,w), y(u,v,w), z(u,v,w)) \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\, dv\, dw$$

eg Cylindrical coordinates: Jacobian again is r. Why?

Geometrically

piece of volume is $dV = r\,dr\,d\theta\,dz$

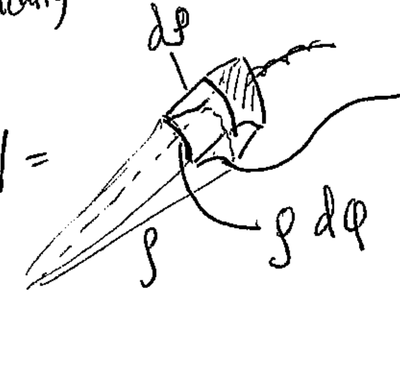
Analytically – take $DT(r,\theta,z)$

eg Spherical coordinates: Jacobian

Analytically, use formulas take $DT(\rho, \varphi, \theta)$

Geometrically

$dV =$



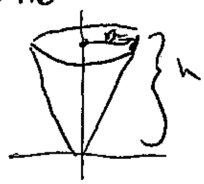
comes from rotating around z-axis at distance r from the z axis so

$r\,d\theta$.

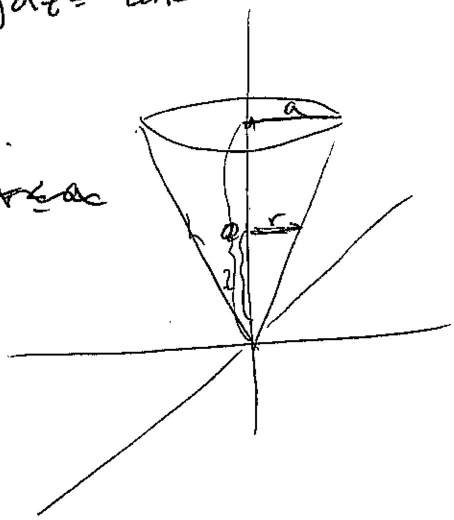
But $r = \rho \sin\varphi$, so

$dV = \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$.

e.g's Find the volume of a cone both with spherical and cylindrical coordinates Let



$\iiint_W 1 \, dx\,dy\,dz =$ cone volume



~~$0 \le r \le a$~~

$? \le z \le h$

$0 \le r \le a$

$0 \le \theta \le 2\pi$

$\frac{z}{h} = \frac{r}{a} \Rightarrow z = \frac{rh}{a}$

So

$\frac{rh}{a} \le z \le h.$

$$\int_0^{2\pi} \int_0^a \int_{\frac{rh}{a}}^h 1 \, dV = \int_0^{2\pi} \int_0^a \int_{\frac{rh}{a}}^h r \, dz \, dr \, d\theta$$

$$\cancel{\int_0^{2\pi} \int_0^a \frac{1}{2}h^2 - \frac{1}{2}\left(\frac{r^2h^2}{a^2}\right)}$$

$$= \int_0^{2\pi} \int_0^a rz \Big|_{\frac{rh}{a}}^h \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^a rh - \frac{r^2h}{a} \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^2h}{2} - \frac{r^3h}{3a} \Big|_0^a \, d\theta$$

$$= \int_0^{2\pi} \frac{a^2h}{2} - \frac{a^3h}{3a} \, d\theta$$

$$= \int_0^{2\pi} \frac{3a^2h}{6} - \frac{2a^2h}{6} \, d\theta = \int_0^{2\pi} \frac{a^2h}{6} \, d\theta$$

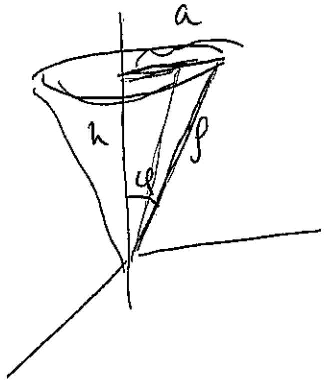
$$= \boxed{\frac{a^2h\pi}{3}}$$

Spherical coordinates

$0 \le \theta \le 2\pi$

$0 \le \rho \le ?$

$0 \le \varphi \le \tan^{-1}\left(\frac{a}{h}\right)$



$\tan\varphi = \frac{a}{h}$

as φ changes

$\cos\varphi = \frac{h}{\rho}$

$\rho = h\sec\phi$ is the plane

Now

$$\iiint_{cone} 1\, dV = \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{h\sec\varphi} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \frac{h^3\sec^3\varphi}{3} \sin\varphi \, d\varphi\, d\theta$$

$$= \frac{h^3}{3}\int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \tan\varphi \sec^2\varphi \, d\varphi\, d\theta$$

$$= \frac{h^3}{3}\int_0^{2\pi} \frac{1}{2}\tan^2\left(\tan^{-1}\left(\frac{a}{h}\right)\right) d\theta$$

$$= \frac{h^3}{3} \cdot \int_0^{2\pi} \frac{1}{2}\left(\frac{a}{h}\right)^2 d\theta = \frac{\pi a^2 h}{3}.$$