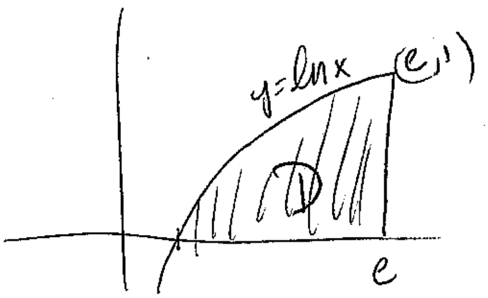
Find this area:



1-D calculus:

$\int_1^e \ln x \, dx \Rightarrow$ integration by parts

$\Rightarrow x\ln x - x \Big|_1^e$

$= e\ln e - e - (\ln 1 - 1)$

$= e - e - 0 + 1 = 1.$

2-D calculus:

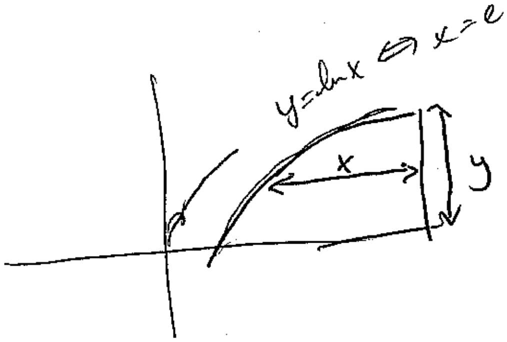
Sometimes taking a 1-D ~~vari~~ integral and making it into a multiple integral helps make the problem easier.

Note: Area of $D = \iint_D 1 \, dA$ (why? geometrically Volume = area of base × height = area of base × 1.)

$$= \int_1^e \int_0^{\ln x} 1 \, dy \, dx$$

$$= \int_1^e \ln x \, dx \ \ldots$$

But, on the other hand, D can also be realized as



$\int_0^1 \int_{e^y}^{e} 1 \, dx\, dy = \int_0^1 e - e^y \, dy$

$= ey - e^y \Big|_0^1$

$= (e - e) - (0 - 1)$

$= 1.$

It doesn't seem than really advantage has been gained. But, in general, suppose D is of type 3, i.e., it's type 1 and type 2.

Type 1 $\Rightarrow$ $a \le x \le b$ $\quad \delta(x) \le y \le \gamma(x)$

Type 2 $\Rightarrow$ $c \le y \le d$ $\quad \alpha(y) \le x \le \beta(y)$.

So

$$\iint_D f(x,y)\,dA = \int_a^b \int_{\delta(x)}^{\gamma(x)} f(x,y)\,dy\,dx = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx\,dy$$

and we say we can ~~switch~~ change the order of integration.

E.g. By changing the order of integration, evaluate

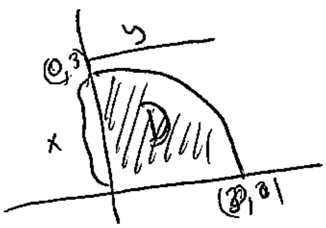
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-y^2}\,dy\,dx .$$

At best, the integral as it stands can be evaluated using trig substitution.

Let's avoid that,

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-y^2}\,dy\,dx = \iint_D \sqrt{9-y^2}\,dy\,dx$$

where



So D also is the set of (x,y) s.t.

$0 \le y \le 3 \qquad 0 \le x \le \sqrt{9-y^2}$

Thus

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9-y^2}\, dy\, dx = \int_0^3 \int_0^{\sqrt{9-y^2}} \sqrt{9-y^2}\, dx\, dy$$

$$= \int_0^3 x\sqrt{9-y^2}\Big|_{x=0}^{x=\sqrt{9-y^2}} dy$$

$$= \int_0^3 \sqrt{9-y^2}\sqrt{9-y^2} - 0\, dy$$

$$= \int_0^3 (9-y^2)\, dy$$

$$= 9y - \frac{y^3}{3}\Big|_0^3 = 27-9 = 18.$$

In short, switching the order of integration may make an integral ~~po~~ easier.

e.g Even computers need ~~the~~ a change in the order of integration:

$\int_0^1 \int_{\sin^{-1} y}^{\pi/2} e^{\cos x}\, dx\, dy$ — Maple 9.5 can't do

Change the order — maple 9.5 can do.

Triple integrals

Double $\Leftrightarrow$ triple dictionary

Double

integral over a rectangle R $[a,b]\times[c,d]$

break it up into ~~squares~~ rectangles of area ΔA_{ij}

$$\iint_R f(x,y)\,dA = \lim_{\substack{\Delta x_i,\\ \Delta y_j \to 0}} \sum_{i,j=1}^{n} f(c_{ij})\,\Delta A_{ij}$$

Thm If f is bounded on R and the set of discontinuities has area 0, then $\iint_R f\,dA$ exists

Fubini's Thm double integrals $\Leftrightarrow$ iterated integrals

elementary regions in $\mathbb{R}^2$

Type 1

Type 2

Type 3

Triple

integral over a box B $[a,b]\times[c,d]\times[p,q]$

break it up into boxes of volume ΔV_{ijk}

$$\iiint_B f(x,y,z)\,dV = \lim_{\substack{\Delta x_i \to 0\\ \Delta y_j \to 0\\ \Delta z_k \to 0}} \sum_{i,j,k=1}^{n} f(c_{ijk})\,\Delta V_{ijk}$$

Thm If f is bounded on B and the set of discontinuities of f on B has ~~zero~~ volume 0, then $\iiint_B f\,dV$ exists.

Fubini's Theorem:

$$\iiint_B f\,dV = \int_a^b \int_c^d \int_p^q f\;dz\,dy\,dx$$

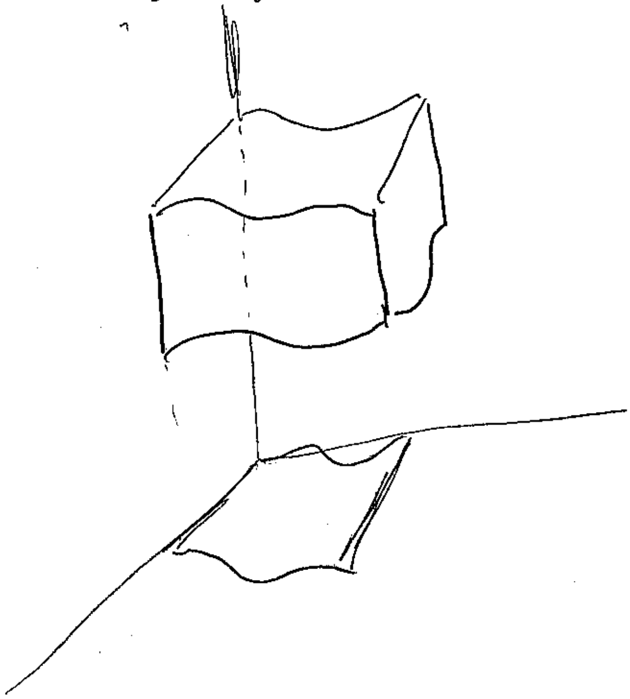
$$= \int_a^b \int_p^q \int_c^d f\;dy\,dz\,dx$$

$$= \text{etc.}$$

elementary regions in $\mathbb{R}^3$.

~~Type~~

elementary region W in $\mathbb{R}^3$:



Type I: If W is the set of all (x,y,z)

s.t. $c \le y \le d$

$\alpha(y) \le x \le \beta(y)$

$\varphi(x,y) \le z \le \psi(x,y)$

[i.e., if (x,y) live in a planar region of type I and z is bounded above & below by a surface.]

OR

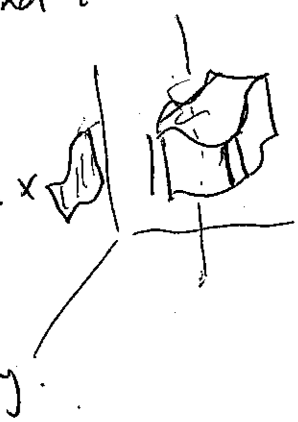
If W is the set of all (x,y,z)

s.t. $a \le x \le b$

$\gamma(x) \le y \le \delta(x)$

$\varphi(x,y) \le z \le \psi(x,y)$

[i.e., if (x,y) live in a planar region of type 2, and z is bounded by two surfaces]

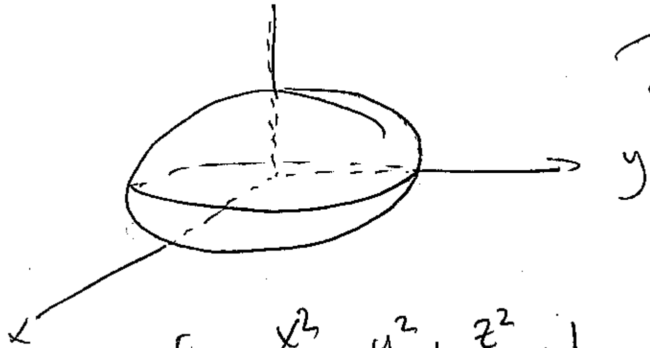


Type 2: Same as type I but change the roles of z & x

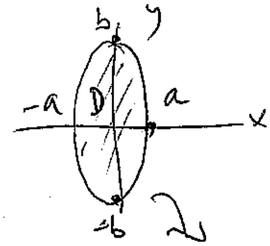
Type 3: Same as type I but change role of z & y.

Type 4: All types at once

E.g



shadow

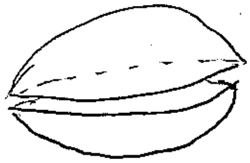


$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$E: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

So $W = \left\{ -\sqrt{c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)} \le z \le \sqrt{c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)},\ -b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}},\ -a \le x \le a \right\}.$

So type I



Also type II

Also type II

$$-a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}} \le x \le a\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}} \qquad -b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}-\frac{z^2}{c^2}}$$

Like before define $f^{ext}(x,y,z) = \begin{cases} f(x,y,z) & (x,y,z)\in W \\ 0 & (x,y,z)\notin W \end{cases}$

Then $\iiint_W f\,dV = \int\iint_B f^{ext}\,dV$.

E.g if type 1,

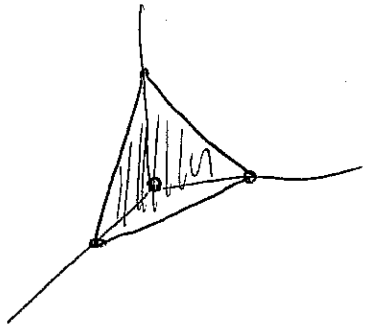
$$\iiint_W f\,dV = \int_a^b \int_{\gamma(x)}^{\delta(x)} \int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

compare

etc.

~~See how~~ ~~dsm~~

W = tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$



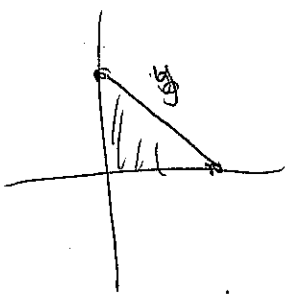
Suppose the mass density at $(x,y,z) = 1+xy$.
How much does the tetrahedron weigh?

z is bounded between 0 and the plane through $(1,0,0)$ $(0,1,0)$ & $(0,0,1)$, $P: x+y+z=1$

So

$$0 \le z \le 1-x-y$$

y is bounded by the line through $(1,0)$ & $(0,1)$



$$0 \le y \le 1-x$$

x is bounded by 0 and 1.

$$0 \le x \le 1.$$ So

$$\iiint_W \underbrace{1+xy}_{g/m^3} \ \underbrace{dV}_{m^3} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1+xy \ \underbrace{dz\,dy\,dx}_{\text{why?}}$$

$$= \int_0^1 \int_0^{1-x} (1+xy)z \Big|_{z=0}^{z=1-x-y} dy\,dx$$

$$= \int_0^1 \int_0^{1-x} (1+xy)(1-x-y)\, dy\,dx$$

$$= \int_0^1 \int_0^{1-x} 1-x-y+xy-x^2y-xy^2 \, dy\,dx$$

$$\vdots$$

$$= \frac{7}{40}.$$