A Comment on Iterated Integrals

Today (Monday, September 21), we looked at the integral

$$V := \iint_R x e^{xy}\, dA$$

where R is the rectangle $[0,2] \times [0,1]$. At first blush, this seemed easy. As we saw in lecture, Fubini's Theorem says that

$$\begin{aligned}
\iint_R x e^{xy}\, dA &= \int_0^2 \int_0^1 x e^{xy}\, dy\, dx \\
&= \int_0^2 \Big(e^{xy} \Big|_{y=0}^{y=2} \Big)\, dx \\
&= \int_0^2 (e^x - 1)\, dx \\
&= e^2 - 3.
\end{aligned}$$

The puzzling thing was that Fubini's Theorem also tells us that

$$\iint_R x e^{xy}\, dA = \int_0^1 \int_0^2 x e^{xy}\, dx\, dy.$$

To do the first integral, we used integration by parts with $u = x$ and $dv = e^{xy} dx$. Then $du = dx$ and $v = \frac{1}{y} e^{xy}$. This resulted in

$$\begin{aligned}
\iint_R x e^{xy}\, dA &= \int_0^1 \int_0^2 x e^{xy}\, dx\, dy. \\
&= \int_0^1 \Big(\frac{2e^{2y}}{y} - \frac{e^{2y}}{y^2} + \frac{1}{y^2} \Big)\, dy \qquad (1)
\end{aligned}$$

Unfortunately, it is not at all clear how to find an anti-derivative for the integrand in (1). We know from Fubini's Theorem that the answer is $e^2 - 3$, but it does not seem fair that we can't work it out.

It turns out that we can do the integral, but we need a bit of hard work. (What follows is very similar to what is done in Example 3 of §15.2 of our text.)

Integration by parts tells us that

$$\int \frac{2e^{2y}}{y}\, dy = \frac{e^{2y}}{y} + \int \frac{e^{2y}}{y^2}\, dy.$$

Written another way,

$$\int \frac{2e^{2y}}{y}\,dy - \int \frac{e^{2y}}{y^2}\,dy = \frac{e^{2y}}{y}.$$

Since these are indefinite integrals, this just means that any antiderivative of $\frac{2e^{2y}}{y}$ differs from an antiderivative of $\frac{e^{2y}}{y^2}$ by $\frac{e^{2y}}{y}$ (plus a constant). Consequently,

$$\begin{aligned}
\int_0^1 \Big(\frac{2e^{2y}}{y} - \frac{e^{2y}}{y^2} + \frac{1}{y^2}\Big)\,dy &= \Big(\int \frac{2e^{2y}}{y}\,dy - \int \frac{e^{2y}}{y^2}\,dy - \frac{1}{y}\Big)\Big|_0^1 \\
&= \Big(\frac{e^{2y}}{y} - \frac{1}{y}\Big)\Big|_0^1 \\
&= \Big(\frac{e^{2y}-1}{y}\Big)\Big|_0^1 \\
&= e^2 - 1 - \lim_{y\to 0}\frac{e^{2y}-1}{y} \\
&= e^2 - 3.
\end{aligned}$$

Of course, as we observed in class, it is just way easier to use the first method.