Math 13 Fall 2004
Calculus of Vector-valued Functions

Example of a function that has both partial derivatives at (0, 0), but is not differentiable there

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } x = y = 0. \end{cases}$$

One can check that f is continuous everywhere in $\mathbb{R}^2$. We can compute its both partial derivatives at $(0,0)$ explicitly:

$$f_x(0,0) = \frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)}{x} = \lim_{x\to 0} \frac{x^3}{(x^2+0^2)x} = \lim_{x\to 0} 1 = 1;$$

$$f_y(0,0) = \frac{\partial f}{\partial x}(0,0) = \lim_{y\to 0} \frac{f(0,y)}{y} = \lim_{y\to 0} \frac{0^3}{(0^2+y^2)y} = \lim_{y\to 0} 0 = 0.$$

So the linear approximation of f at $(0,0)$ would be $h(x,y) = x$. Let's check how good it is:

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - h(x,y)}{||(x,y)-(0,0)||} = \lim_{(x,y)\to(0,0)} \frac{\frac{x^3}{x^2+y^2} - x}{\sqrt{x^2+y^2}} = -\lim_{(x,y)\to(0,0)} \frac{xy^2}{\sqrt{x^2+y^2}(x^2+y^2)}$$

But this limit does **not** exist. Indeed, if $x = 0$, it should be 0, but if $y = x$, it should be -1 for $x > 0$ and 1 for $x < 0$.

The function f is not differentiable at $(0,0)$.

The reason is that its partial derivatives are both **not** continuous in a neighborhood of $(0,0)$:

$$f_x(x,y) = \frac{\partial f}{\partial x}(x,y) = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} \qquad \text{and} \qquad f_y(x,y) = \frac{\partial f}{\partial y}(x,y) = -\frac{2x^3y}{(x^2+y^2)^2}.$$

The limits of both f_x and f_y DNE as $(x,y) \to (0,0)$. If $x = 0$, they should be both 0, but if $y = x$, they should be 1 and $-1/2$, respectively.