Math 13 Fall 2004

Calculus of Vector-valued Functions

Example of a function that has different mixed partial derivatives at (0,0)

October 4, 2004

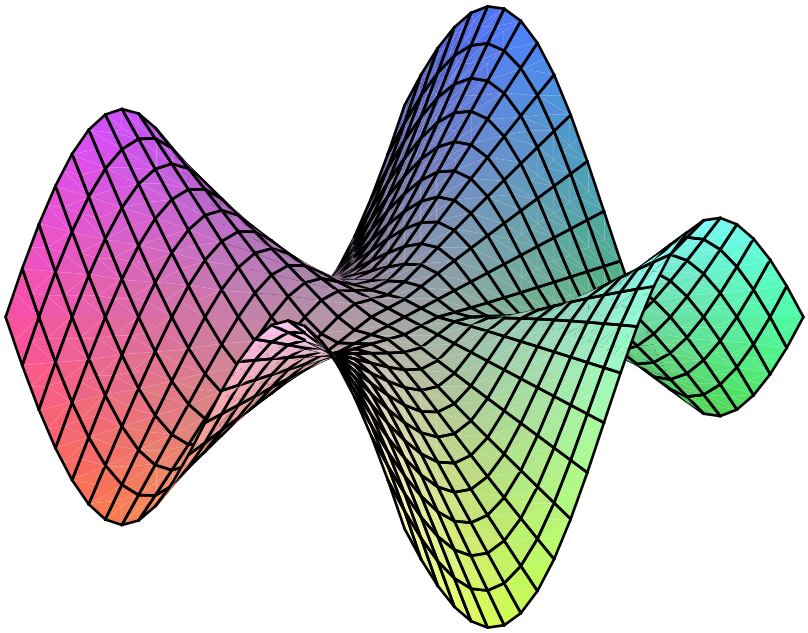
Define a scalar-valued function of two variables

```
> f := (x, y) -> x * y * (x^2 - y^2) / (x^2 + y^2);
```

$$f := (x, y) \to \frac{x\,y\,(x^2 - y^2)}{x^2 + y^2}$$

Have a look at its graph

```
> plot3d(f(x, y), x = -1..1, y = -1..1);
```



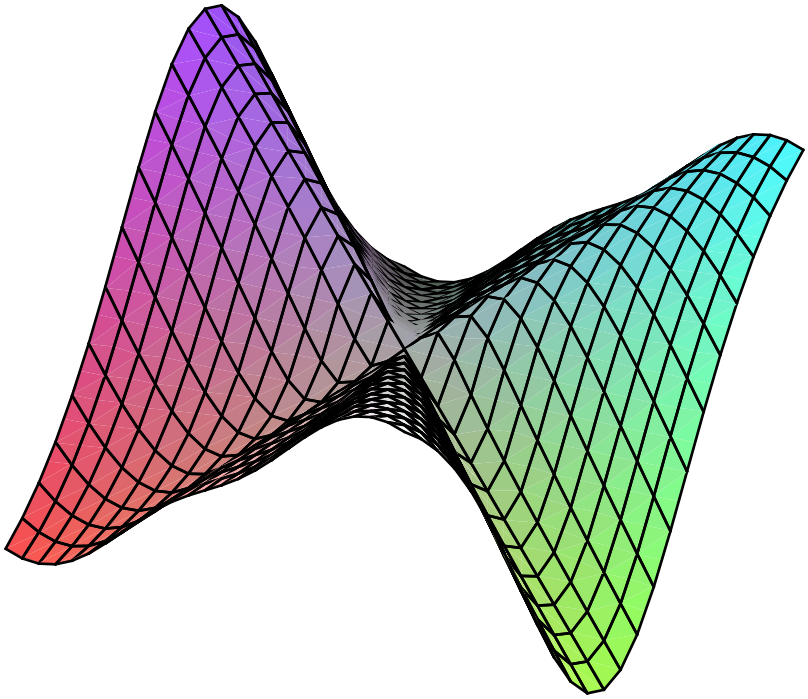
Both partial derivatives of f are continuous everywhere, so f is differentiable at (0, 0)

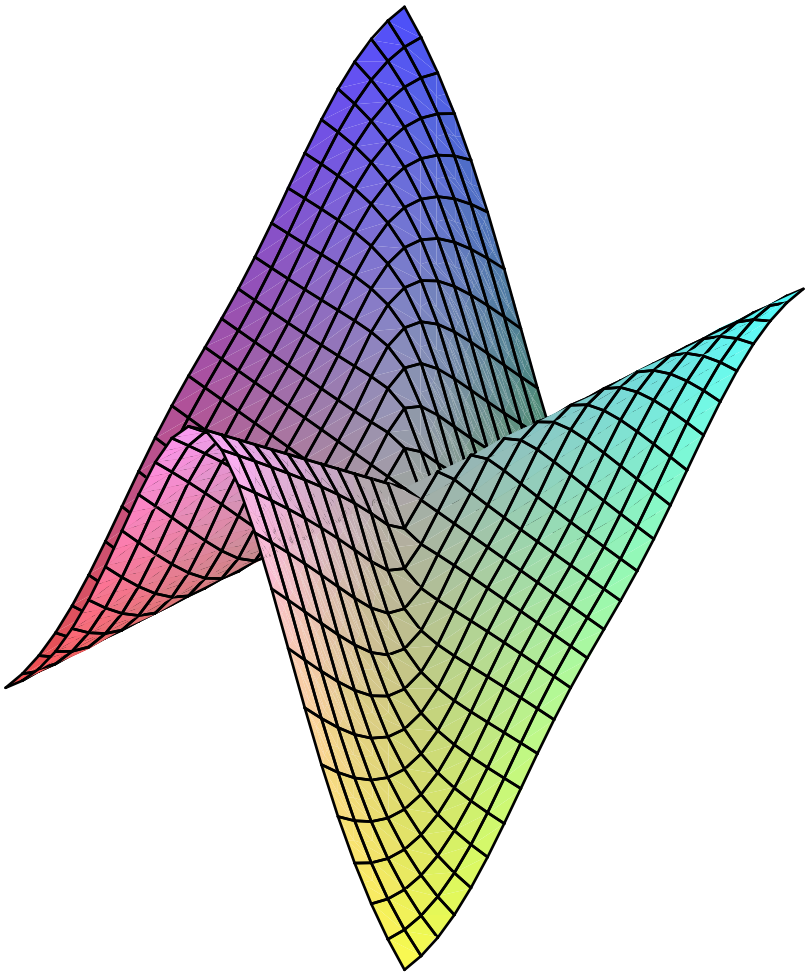
```
> f_x := factor(diff(f(x, y), x));
  f_y := factor(diff(f(x, y), y));
```

$$f\_x := \frac{y\,(x^4 - y^4 + 4\,x^2\,y^2)}{(x^2 + y^2)^2}$$

$$f\_y := \frac{x\,(x^4 - y^4 - 4\,x^2\,y^2)}{(x^2 + y^2)^2}$$

```
> plot3d(f_x, x = -2..2, y = -2..2);
  plot3d(f_y, x = -2..2, y = -2..2);
```





Let's explicitly compute the mixed partial derivatives of f at (0, 0)

```
> f_xy_00 := limit(subs(x = h, y = 0, f_x) / h, h = 0);
  f_yx_00 := limit(subs(x = 0, y = h, f_x) / h, h = 0);
```

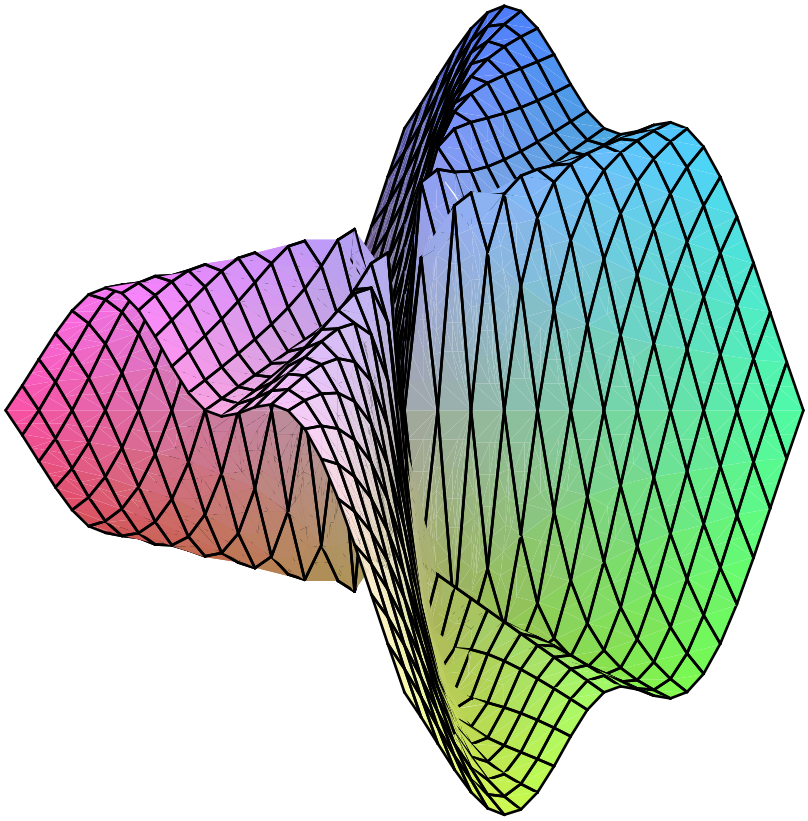
$$f\_xy\_00 := 0$$

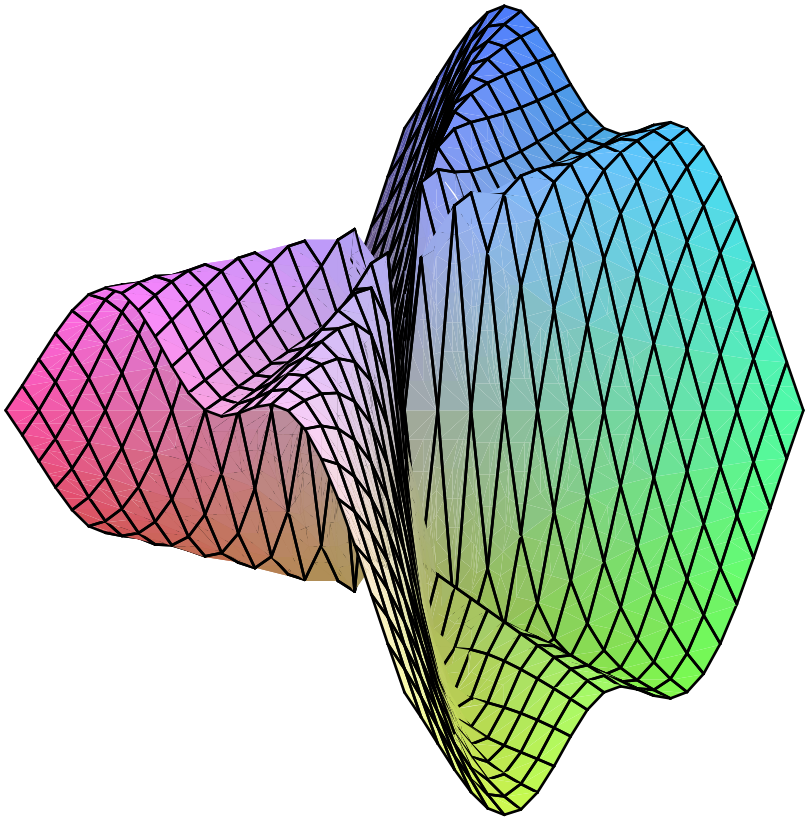
$$f\_yx\_00 := -1$$

They are **different!!!**

Let's plot both mixed partial derivatives of f

```
> plot3d(diff(f(x, y), y, x), x = -1..1, y = -1..1);
  plot3d(diff(f(x, y), x, y), x = -1..1, y = -1..1);
```





They are obviously discontinuous!!!

Remark: mixed partial derivatives are the same away from (0, 0)

```
> factor(diff(f(x, y), y, x));
  factor(diff(f(x, y), x, y));
```

$$\frac{(-y+x)(x+y)(y^4+10x^2y^2+x^4)}{(x^2+y^2)^3}$$

$$\frac{(-y+x)(x+y)(y^4+10x^2y^2+x^4)}{(x^2+y^2)^3}$$

```
>
```