Math 13 Fall 2004
Calculus of Vector-valued Functions

Example of a function that has different mixed partial derivatives at (0, 0)

Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } x = y = 0. \end{cases}$$

The partial derivatives of f are given by

$$f_x(x,y) = \begin{cases} \dfrac{y(x^4-y^4-4x^2y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } x = y = 0 \end{cases}$$

and

$$f_y(x,y) = \begin{cases} \dfrac{x(x^4-y^4-4x^2y^2)}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } x = y = 0. \end{cases}$$

One can check that f as well as f_x and f_y are continuous everywhere in $\mathbb{R}^2$. Hence, f is differentiable in $\mathbb{R}^2$.

We can compute the second order mixed partial derivatives of f at $(0,0)$ explicitly:

$$f_{yx}(0,0) = \frac{\partial}{\partial x}(f_y)(0,0) = \lim_{x\to 0} \frac{f_y(x,0)}{x} = \lim_{x\to 0} \frac{x(x^4-0^4-4x^2 0^2)}{(x^2+0^2)^2 x} = \lim_{x\to 0} 1 = 1;$$

$$f_{xy}(0,0) = \frac{\partial}{\partial y}(f_x)(0,0) = \lim_{y\to 0} \frac{f_x(0,y)}{x} = \lim_{y\to 0} \frac{y(0^4-y^4-4(0^2y^2))}{(0^2+y^2)^2 y} = \lim_{y\to 0} -1 = -1.$$

The mixed partial derivatives $f_{yx}(0,0)$ and $f_{xy}(0,0)$ of f at $(0,0)$ are different

The reason is that they are both **not** continuous in a neighborhood of $(0,0)$:

$$f_{yx}(x,y) = \frac{\partial}{\partial x}(f_y)(x,y) = \frac{(x^2-y^2)(x^4+10x^2y^2+y^4)}{(x^2+y^2)^3} = \frac{\partial}{\partial y}(f_x)(x,y) = f_{xy}(x,y).$$

The limits of both f_{yx} and f_{xy} DNE as $(x,y) \to (0,0)$. If $x = 0$, they should be -1, but if $y = 0$, they should be 1.

Notice that $f_{xy}(x,y) = f_{yx}(x,y)$ away from the origin. Indeed, all the derivatives of f are continuous there.