Major Facts about Limits and Continuity

Fact 1. **(Uniqueness of limits)** If a limit exists, it is unique:

If $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$, $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{L}$, and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{M}$, **then** $\mathbf{L} = \mathbf{M}$.

Fact 2. **(Algebraic properties of limits)**

Let $\mathbf{F}, \mathbf{G} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f, g : X \subset \mathbb{R}^n \to \mathbb{R}$. Let also $k \in \mathbb{R}$.

1. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{L}$ and $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{G}(\mathbf{x}) = \mathbf{M}$, **then** $\lim_{\mathbf{x}\to\mathbf{a}} (\mathbf{F} + \mathbf{G})(\mathbf{x}) = \mathbf{L} + \mathbf{M}$.

2. If $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{L}$, **then** $\lim_{\mathbf{x}\to\mathbf{a}} k\mathbf{F}(\mathbf{x}) = k\mathbf{L}$.

3. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M$, **then** $\lim_{\mathbf{x}\to\mathbf{a}} (fg)(\mathbf{x}) = LM$.

4. If $\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$, $g(\mathbf{x}) \neq 0$ for $\mathbf{x} \in X$ and $\lim_{\mathbf{x}\to\mathbf{a}} g(\mathbf{x}) = M \neq 0$, **then** $\lim_{\mathbf{x}\to\mathbf{a}} (f/g)(\mathbf{x}) = L/M$.

Fact 3. Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$. Then $\lim_{\mathbf{x}\to\mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{L}$ **if and only if** $\lim_{\mathbf{x}\to\mathbf{a}} F_i(\mathbf{x}) = L_i$ for all $i = 1, 2, \ldots, m$, **where** $\mathbf{F} = (F_1, F_2, \ldots, F_m)$ and $\mathbf{L} = (L_1, L_2, \ldots, L_m)$.

Fact 4. Let $\mathbf{F}, \mathbf{G} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $f, g : X \subset \mathbb{R}^n \to \mathbb{R}$. Let also $k \in \mathbb{R}$.

1. If $\mathbf{F}$ and $\mathbf{G}$ are continuous at $\mathbf{a} \in X$, **then** $\mathbf{F} + \mathbf{G}$ is continuous at $\mathbf{a}$.

2. If $\mathbf{F}$ is continuous at $\mathbf{a} \in X$, **then** $k\mathbf{F}$ is continuous at $\mathbf{a}$.

3. If f and g are continuous at $\mathbf{a} \in X$, **then** fg is continuous at $\mathbf{a}$.

4. If f and g are continuous at $\mathbf{a} \in X$ and $g \neq 0$, **then** f/g is continuous at $\mathbf{a}$.

5. $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $\mathbf{a} \in X$ **if and only if** all $F_i : X \subset \mathbb{R}^n \to \mathbb{R}$ are continuous at $\mathbf{a}$, **where** $\mathbf{F} = (F_1, F_2, \ldots, F_m)$.

Fact 5. **(Composition of continuous functions)** If $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{G} : Y \subset \mathbb{R}^m \to \mathbb{R}^p$ are continuous and Range$(\mathbf{F}) \subset Y$, **then** $\mathbf{G} \circ \mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^p$ is also continuous.

MAJOR FACTS ABOUT DERIVATIVES

FACT 1. If $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in X$, **then** it is continuous at $\mathbf{a}$.

FACT 2. Let $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ such that all $\dfrac{\partial F_i}{\partial x_j}$ exist and are continuous in a neighborhood of $\mathbf{a} \in X$. **Then** $\mathbf{F}$ is differentiable at $\mathbf{a}$.

FACT 3. $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $\mathbf{a} \in X$ **if and only if** all $F_i : X \subset \mathbb{R}^n \to \mathbb{R}$ are differentiable at $\mathbf{a}$, **where** $\mathbf{F} = (F_1, F_2, \ldots, F_m)$.

FACT 4. **(Linearity of the derivative)**
Let $\mathbf{F}, \mathbf{G} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at $\mathbf{a} \in X$ and $k \in \mathbb{R}$. **Then**

1. $\mathbf{F} + \mathbf{G}$ is differentiable at $\mathbf{a}$ and $D(\mathbf{F} + \mathbf{G})(\mathbf{a}) = D\mathbf{F}(\mathbf{a}) + D\mathbf{G}(\mathbf{a})$.

2. $k\mathbf{F}$ is differentiable at $\mathbf{a}$ and $D(k\mathbf{F})(\mathbf{a}) = kD\mathbf{F}(\mathbf{a})$.

FACT 5. Let $f, g : X \subset \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in X$. **Then**

1. fg is differentiable at $\mathbf{a}$ and $D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a})$.

2. **if** $g \neq 0$, f/g is differentiable at $\mathbf{a}$ and $D(f/g)(\mathbf{a}) = \dfrac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$.

FACT 6. **(The chain rule)** If $\mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{G} : Y \subset \mathbb{R}^m \to \mathbb{R}^p$ are differentiable at $\mathbf{a}$ and $\mathbf{b} = \mathbf{F}(\mathbf{a})$, respectively, and $\text{Range}(\mathbf{F}) \subset Y$, **then** $\mathbf{G} \circ \mathbf{F} : X \subset \mathbb{R}^n \to \mathbb{R}^p$ is also differentiable at $\mathbf{a}$ and $D(\mathbf{G} \circ \mathbf{F})(\mathbf{a}) = D\mathbf{G}(\mathbf{b})D\mathbf{F}(\mathbf{a})$.