Major Facts about Double Integrals

Fact 1. **(Definition)** $\displaystyle\iint\limits_R f\,dA = \lim_{\Delta x_i, \Delta y_j \to 0} \sum_{i,j=1}^{n} f(\mathbf{c}_{ij})\Delta x_i \Delta y_j.$

Fact 2. **<u>If</u>** f is continuous on the closed rectangle R, **<u>then</u>** $\displaystyle\iint\limits_R f\,dA$ exists.

Fact 3. **<u>If</u>** f is bounded on R and **<u>if</u>** the set of discontinuities of f on R has zero area, **<u>then</u>** $\displaystyle\iint\limits_R f\,dA$ exists.

Fact 4. **(Fubini's Theorem)** **<u>If</u>** f is bounded on $R = [a, b] \times [c, d]$ and all discontinuities of f lie on a union of graphs of continuous functions, **<u>then</u>**

$$\iint\limits_R f\,dA = \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_c^d \int_a^b f(x, y)\,dx\,dy.$$

Fact 5. **(Properties of the integral)** **<u>If</u>** f and g are both integrable of the closed rectangle R, **<u>then</u>**

1. $f + g$ is also integrable on R and $\displaystyle\iint\limits_R (f + g)\,dA = \iint\limits_R f\,dA + \iint\limits_R g\,dA$;
2. cf is also integrable on R for any constant $c \in \mathbb{R}$ and $\displaystyle\iint\limits_R cf\,dA = c\iint\limits_R f\,dA$;
3. **<u>if</u>** $f(x, y) \leq g(x, y)$ for all $(x, y) \in R$, **<u>then</u>** $\displaystyle\iint\limits_R f\,dA \leq \iint\limits_R g\,dA$;
4. $|f|$ is also integrable on R and $\displaystyle\left|\iint\limits_R f\,dA\right| \leq \iint\limits_R |f|\,dA$.

Fact 6. **<u>If</u>** f is a continuous function on an elementary region D in $\mathbb{R}^2$, **<u>then</u>**

1. if D is of **type I**, then $\displaystyle\iint\limits_D f\,dA = \int_a^b \int_{\gamma(x)}^{\delta(x)} f(x, y)\,dy\,dx$;
2. if D is of **type II**, then $\displaystyle\iint\limits_D f\,dA = \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x, y)\,dx\,dy$.