Math 13 Fall 2004
Calculus of Vector-valued Functions
Example of the multi-dimensional chain rule

Consider the following situation: $$\begin{array}{ccccc} \mathbb{R}^2 & \xrightarrow{\ \mathbf{g}\ } & \mathbb{R}^2 & \xrightarrow{\ \mathbf{f}\ } & \mathbb{R}^3 \\ \Cup & & \Cup & & \Cup \\ \mathbf{t} & \longmapsto & \mathbf{x} & \longmapsto & \mathbf{w} \end{array}$$

where $\mathbf{f}(x_1,\, x_2) = \big(x_1x_2,\, \sin(x_1+x_2),\, e^{x_1^2+x_2^2}\big)$ and $\mathbf{g}(t_1,\, t_2) = \big(t_1^2+t_2^2,\, t_1^2-t_2^2\big)$.

Let $\mathbf{h}(\mathbf{t}) = (\mathbf{f}\circ\mathbf{g})(\mathbf{t})$. Then $\mathbf{h}(t_1,\, t_2) = \Big((t_1^2+t_2^2)(t_1^2-t_2^2),\, \sin(t_1^2+t_2^2+t_1^2-t_2^2),\, e^{(t_1^2+t_2^2)^2+(t_1^2-t_2^2)^2}\Big)$

$$= \big(t_1^4-t_2^4,\, \sin(2t_1^2),\, e^{2t_1^4+2t_2^4}\big).$$

Compute the matrix of partial derivatives for $\mathbf{h}$:

$$D\mathbf{h}(\mathbf{t}) = \begin{pmatrix} \frac{\partial h_1}{\partial t_1} & \frac{\partial h_1}{\partial t_2} \\ \frac{\partial h_2}{\partial t_1} & \frac{\partial h_2}{\partial t_2} \\ \frac{\partial h_3}{\partial t_1} & \frac{\partial h_3}{\partial t_2} \end{pmatrix} = \begin{pmatrix} 4t_1^3 & -4t_2^3 \\ 4t_1\cos(2t_1^2) & 0 \\ 8t_1^3e^{2t_1^4+2t_2^4} & 8t_2^3e^{2t_1^4+2t_2^4} \end{pmatrix}.$$

On the other hand:

$$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{pmatrix} = \begin{pmatrix} x_2 & x_1 \\ \cos(x_1+x_2) & \cos(x_1+x_2) \\ 2x_1e^{x_1^2+x_2^2} & 2x_2e^{x_1^2+x_2^2} \end{pmatrix}$$

and

$$D\mathbf{g}(\mathbf{t}) = \begin{pmatrix} \frac{\partial g_1}{\partial t_1} & \frac{\partial g_1}{\partial t_2} \\ \frac{\partial g_2}{\partial t_1} & \frac{\partial g_2}{\partial t_2} \end{pmatrix} = \begin{pmatrix} 2t_1 & 2t_2 \\ 2t_1 & -2t_2 \end{pmatrix}.$$

Then

$$\begin{aligned} D\mathbf{f}(\mathbf{x})D\mathbf{g}(\mathbf{t}) &= \begin{pmatrix} x_2 & x_1 \\ \cos(x_1+x_2) & \cos(x_1+x_2) \\ 2x_1e^{x_1^2+x_2^2} & 2x_2e^{x_1^2+x_2^2} \end{pmatrix} \begin{pmatrix} 2t_1 & 2t_2 \\ 2t_1 & -2t_2 \end{pmatrix} \\ &= \begin{pmatrix} t_1^2-t_2^2 & t_1^2+t_2^2 \\ \cos(2t_1^2) & \cos(2t_1^2) \\ 2(t_1^2+t_2^2)e^{2t_1^4+2t_2^4} & 2(t_1^2-t_2^2)e^{2t_1^4+2t_2^4} \end{pmatrix} \begin{pmatrix} 2t_1 & 2t_2 \\ 2t_1 & -2t_2 \end{pmatrix} \\ &= \begin{pmatrix} 2t_1(t_1^2-t_2^2)+2t_1(t_1^2+t_2^2) & 2t_2(t_1^2-t_2^2)-2t_2(t_1^2+t_2^2) \\ 2t_1\cos(2t_1^2)+2t_1\cos(2t_1^2) & 2t_2\cos(2t_1^2)-2t_2\cos(2t_1^2) \\ \big(4t_1(t_1^2+t_2^2)+4t_1(t_1^2-t_2^2)\big)e^{2t_1^4+2t_2^4} & \big(4t_2(t_1^2+t_2^2)-4t_2(t_1^2-t_2^2)\big)e^{2t_1^4+2t_2^4} \end{pmatrix} \\ &= \begin{pmatrix} 4t_1^3 & -4t_2^3 \\ 4t_1\cos(2t_1^2) & 0 \\ 8t_1^3e^{2t_1^4+2t_2^4} & 8t_2^3e^{2t_1^4+2t_2^4} \end{pmatrix} = D\mathbf{h}(\mathbf{t}). \end{aligned}$$

This verifies the **chain rule** for **f**, **g**, and **h**.