Math 123 *OPTIONAL* Homework Assignment on Nets

Not not turn in.

1. Suppose that X is a first countable space. Show that each $x \in X$ has a neighborhood basis of open sets $\{ U_n \}_{n=1}^{\infty}$ such that $U_{n+1} \subseteq U_n$.

2. Let X and Y be 1^{st} countable spaces.

(a) Show that $\mathcal{O} \subseteq X$ is open in X if and only if every *sequence* converging to some $x \in \mathcal{O}$ is eventually in $\mathcal{O}$.

(b) Show that $F \subseteq X$ is closed if and only if every convergent *sequence* in F converges to a point in F.

(c) Show that $f : X \to Y$ is continuous if and only if whenever $\{ x_n \}_{n=1}^{\infty}$ coverges to $x \in X$ then $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x) \in Y$.

(d)* Let $\{ x_n \}_{n=1}^{\infty}$ be a sequence in X. Show that if $\{ x_n \}_{n=1}^{\infty}$ has a convergent *subnet*, then $\{ x_n \}_{n=1}^{\infty}$ has a convergent *subsequence*. (Hint: if $\{ x_n \}_{n=1}^{\infty}$ has an accumulation point, then it must have a convergent subsequence.)

(e)* Show that 1^{st} countability is required in part (d). (Hint: let ℓ^{∞} denote the set of *bounded* sequences. If $\alpha = \{\alpha_n\}_{n=1}^{\infty} \in \ell^{\infty}$, let I_α be any closed bounded interval in $\mathbf{R}$ such that $\alpha_n \in I_\alpha$ for all n. Set

$$Z = \prod_{\alpha \in \ell^{\infty}} I_\alpha.$$

Consider the sequence $\{ x_n \}_{n=1}^{\infty}$ in the compact space Z defined by $x_n(\alpha) = \alpha_n$.)

ANS: (a) ($\Longleftarrow$) This is easy.
($\Longrightarrow$) Just as in lecture, if $\mathcal{O}$ fails to be open, then there is a $x \in \mathcal{O}$ such that every neighborhood of x intersects $\mathcal{O}^c$. Choose a neighborhood basis at x of the form $\{U_n\}_{n=1}^{\infty}$ with $U_{n+1} \subseteq U_n$. Now choose $x_n \in U_n \cap \mathcal{O}^c$. We have $x_n \to x$ while $\{ x_n \}_{n=1}^{\infty} \subseteq \mathcal{O}^c$, a contradiction.
(d) Let $\{x_{n_\beta}\}_{\beta \in B}$ be a subnet converging to $x \in X$. Let $\{U_n\}$ be a neighborhood basis at x as above.

We will define a subsequence inductively. Since $\{x_{n_\beta}\}$ is eventually in U_1, there is a n_1 so that $x_{n_1} \in U_1$. Now suppose we have choosen $n_1 < n_2 < \cdots < n_k$ with $x_{n_j} \in U_j$ for $j = 1, 2, \ldots, k$. Using the definition of a subnet, there is a $\beta_0 \in B$ so that $\beta \geq \beta_0$ implies that $n_\beta \geq n_k$. On the other hand, since our subnet converges to x, there is a β_0' such that $\beta \geq \beta_0'$ implies that $x_{n_\beta} \in U_{k+1}$. Thus if $\beta \geq \beta_0$ and $\beta \geq \beta_0'$, then we may take $n_{k+1} = n_\beta$. This suffices.

(e) Since Z is compact, $\{ x_n \}_{n=1}^{\infty}$ must have a convergent subnet. Suppose that $\{ x_n \}_{n=1}^{\infty}$ had a convergent subsequence: say $\{ x_{n_k} \}_{k=1}^{\infty}$ converges to $x \in Z$. Now let $\alpha \in \ell^{\infty}$ be defined as follows:

$$\alpha_n = \begin{cases} (-1)^k & \text{if } n = n_k \text{ for some } k \geq 1, \text{ and,} \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $x_{n_k}(\alpha) = (-1)^k$. But we must have $x_{n_k}(\alpha)$ converging to the number $x(\alpha)$. This is a contradiction.

3. Let $\mathcal{H}$ be a separable infinite dimensional (complex) Hilbert space with orthonormal basis $\{ e_n \}_{n=1}^{\infty}$. Let $S = \{ \sqrt{n} e_n \}_{n=1}^{\infty}$, and let W be the weak closure of S in $\mathcal{H}$.

(a) Show that $0 \in W$.

(b) Observe that no *sequence* in S converges weakly to 0.

(Hints: A basic weak neighborhood of 0 is of the form

$$U = \{ h \in \mathcal{H} : |(h \mid k_j)| < \epsilon \text{ for } j = 1, \dots, n \}.$$

If $\sqrt{n} e_n \notin U$, then

$$\sum_{j=1}^{n} |(e_n \mid k_j)|^2 \geq \frac{\epsilon^2}{n}.$$

Observe that this can't happen for all n.

For the second part, observe that the principal of uniform boundedness implies that any weakly convergent *sequence* is bounded. Of course, no bounded sequence from S can converge weakly to 0.)

ANS: As in the hint, if $\sqrt{n} e_n \notin U$, then

$$\sum_{j=1}^{n} |(h \mid k_j)|^2 \geq \frac{\epsilon}{n}. \tag{1}$$

On the other hand,

$$\sum_{j=1}^{n} \|k_j\|^2 = \sum_{j=1}^{n} \sum_{n=1}^{\infty} |(e_n | k_j)|^2 = \sum_{n=1}^{\infty} \sum_{j=1}^{n} |(e_n | k_j)|^2 < \infty.$$

Clearly, the above can't hold if (1) holds for all n. Hence $S \cap U \neq \emptyset$. Since U is arbitrary, we have shown that $0 \in W$.

For the second part, suppose that $h_n \to h$ weakly. Let ϕ_{h_n} be the linear functional $\phi_{h_n}(k) := (k \mid |h_n)$. Then $\|\phi_{h_n}\| = \|h_n\|$. Clearly, $\{ \phi_{h_n}(k) : n \in \mathbf{N} \}$ is bounded for each k. Then the principal of uniform boundedness implies that $\{ \|h_n\| : n \in \mathbf{N} \}$ is bounded. It then follows that if $\{ h_n \}$ is a sequence in S converging to 0 weakly, then there is a M such that $\|h_n\| \leq M$ for all n. This is clearly nonsense. So no sequence from S can converge weakly to 0.