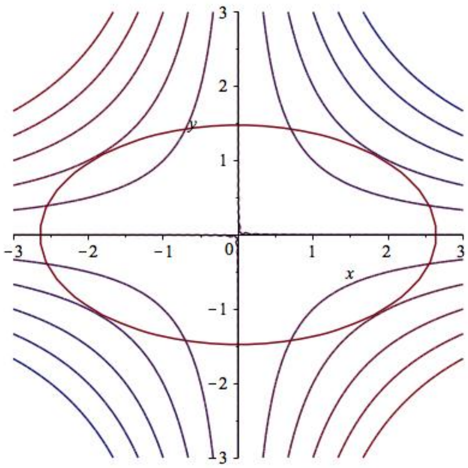
Math 11
Fall 2016
Section 1
Monday, October 10, 2016

First, some important points from the last class:



Finding the largest or smallest value of $f(x_1, \ldots, x_1)$ is called an optimization problem. Finding the largest or smallest value of $f(x_1, \ldots, x_n)$ when $(x_1, \ldots, x_n)$ is required to satisfy some condition (for example, $x^2 + y^2 = 1$) is called a constrained optimization problem, and the condition is the constraint.

When we are trying to maximize or minimize f on a closed, bounded, region, looking at the edge of that region generally involves constraints of the form $g(x_1, \ldots, x_n) = k$ (for example, $x^2 + y^2 = 1$). In other words, $(x_1, \ldots, x_n)$ must lie on some level set (level curve, level surface, $\ldots$) of g.

The method of Lagrange multipliers is designed to solve exactly this kind of problem.

Theorem (the method of Lagrange multipliers): Suppose $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are differentiable functions, and S is a level set of g, defined by $g(x_1, \ldots, x_n) = k$.

If $f(x_1, \ldots, x_n)$ has a largest (or smallest) value on S, then it attains that extreme value at a point $(x_1, \ldots, x_n)$ at which either

$$\nabla g(x_1, \ldots, x_n) = \vec{0}$$

or, for some scalar λ,

$$\nabla f(x_1, \ldots x_n) = \lambda \nabla g(x_1, \ldots, x_n).$$

This means that to solve this problem, we should look for solutions to

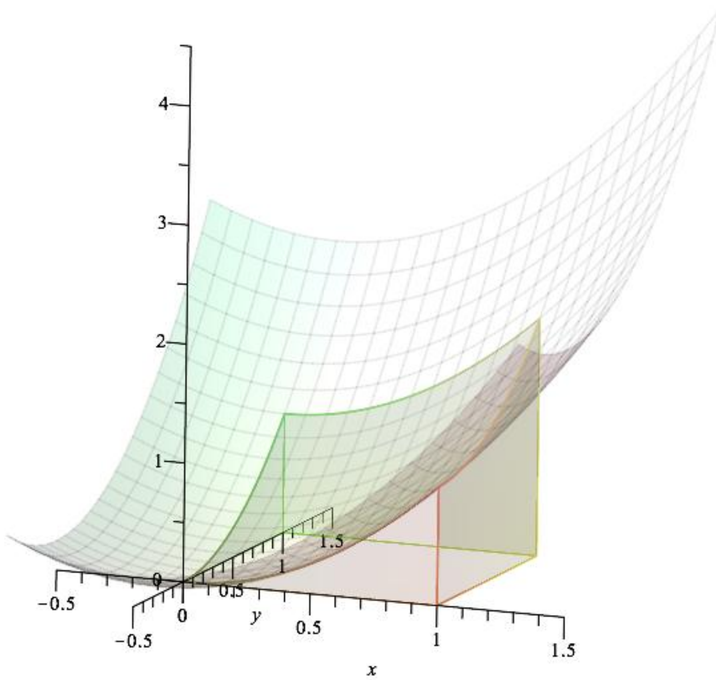
$$\nabla g(x_1, \ldots, x_n) = \vec{0} \quad \& \quad g(x_1, \ldots, x_n) = k$$

and to

$$\nabla f(x_1, \ldots, x_n) = \lambda \nabla g(x_1, \ldots, x_n) \quad \& \quad g(x_1, \ldots, x_n) = k.$$

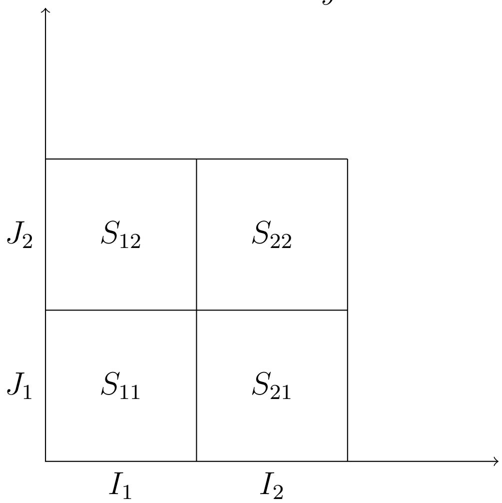
Today we look at integrating functions $f(x, y)$.

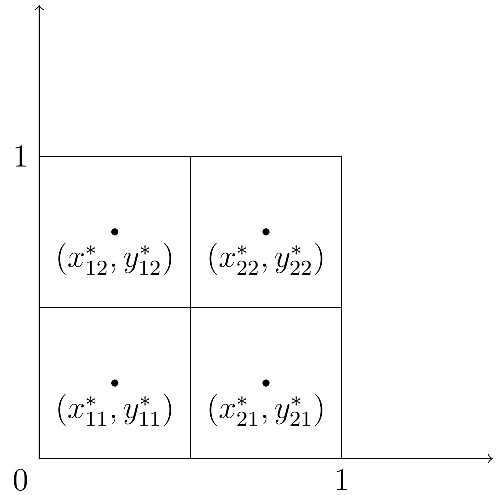
Example: Find the volume of the region lying above the square $0 \le x \le 1$, $0 \le y \le 1$ and below the graph of the function $f(x, y) = x^2 + y^2$.



Method 1: Approximate the volume by adding up the volumes of many skinny rectangular columns, in the same we we approximated the area under a curve by adding up the areas of many skinny rectangles. Take a limit as the number of subdivisions approaches infinity.

For this example, as a not very close approximation, we divide the intervals $0 \le x \le 1$ and $0 \le y \le 1$ into two subintervals (intervals I_1, I_2 and J_1, J_2) of lengths $\Delta x = .5$ and $\Delta y = .5$. This divides the square into four subsquares (squares $S_{11}, S_{12}, S_{21}, S_{22}$) each of which has area $\Delta A = \Delta x \Delta y = .25$.





Approximate the volume under the surface above square i,j by the volume of a rectangular column whose height is the same as the height of the surface $z = f(x, y)$ above some point (x^*_{ij}, y^*_{ij}) in square i,j. The volume of column i,j is

$$f(x^*_{ij}, y^*_{ij})\Delta x \Delta y = ((x^*_{ij})^2 + (y^*_{ij})^2)(.25).$$

We add the volumes of all the columns to get our approximation.

For our example, we will take as our (x^*_{ij}, y^*_{ij}) the midpoints of our small squares, which are $(.25, .25), (.25, .75), (.75, .25), (.75, .75)$. This gives

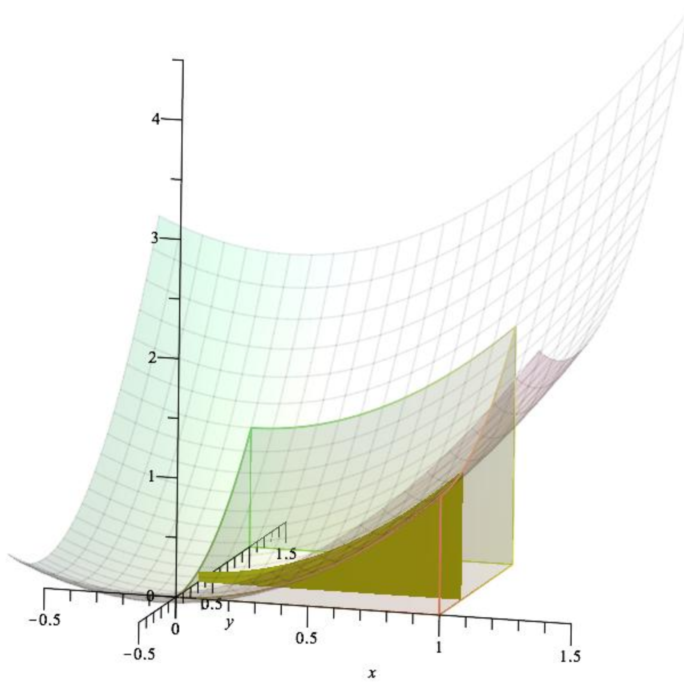
$$V \approx \big((.25)^2 + (.25)^2\big)(.25) + \big((.25)^2 + (.75)^2\big)(.25) +$$

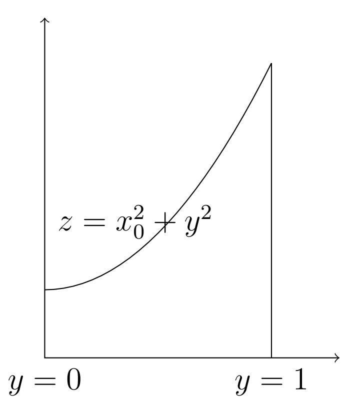
$$\big((.75)^2 + (.25)^2\big)(.25) + \big((.75)^2 + (.75)^2\big)(.25) = .625.$$

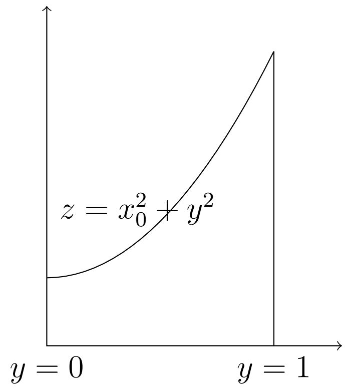
Method 2: Use volumes by slicing. If slicing our region perpendicular to the x-axis where $x = x_0$ yields a slice of area $A(x_0)$, then the volume is

$$V = \int_0^1 A(x)\,dx.$$

Pictured below is the cross-section at $x = x_0$:







From this picture we have $A(x_0) = \int_0^1 x_0^2 + y^2\,dy$. Therefore, the volume of the solid is

$$V = \int_0^1 A(x)\,dx = \int_0^1 \left[\int_0^1 x^2 + y^2\,dy\right] dx.$$

To compute this, keep in mind that in the inner integral, we are integrating with respect to y, and x is playing the role of a constant:

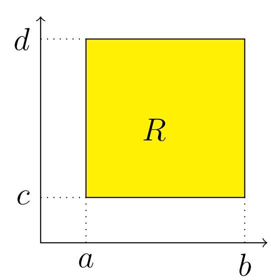
$$\int_0^1 \left[\int_0^1 x^2 + y^2\,dy\right] dx = \int_0^1 \left[yx^2 + \frac{y^3}{3}\right]\Big|_{y=0}^{y=1} dx = \int_0^1 \left[x^2 + \frac{1}{3}\right] dx = \left[\frac{x^3}{3} + \frac{x}{3}\right]\Big|_{x=0}^{x=1} = \frac{2}{3}.$$

This example illustrates integrating a function $f(x, y)$ over a rectangle in the xy-plane. Method 1 leads to the definition of double integral, and method 2 leads to a way to compute double integrals.

First, a piece of notation:

$$R = \{(x, y) \mid a \le x \le b \;\&\; c \le y \le d\} = [a, b] \times [c, d]$$

denotes the rectangle in $\mathbb{R}^2$ whose projection on the horizontal (x) axis is the interval $[a, b]$ and whose projection on the vertical (y) axis is $[c, d]$.



Method 1 leads to the definition of double integral:

Definition: If $R = [a, b] \times [c, d]$ is a rectangle in the xy plane, and $f(x, y)$ is a function whose domain includes R, then

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A,$$

if this limit exists. The notation means the following:

The intervals $[a, b]$ and $[c, d]$ are divided into m subintervals of length $\Delta x = \dfrac{b-a}{m}$ and n subintervals of length $\Delta y = \dfrac{d-c}{n}$, respectively. This divides R into mn-many rectangles of area $\Delta A = \Delta x \Delta y$.

The point (x_{ij}^*, y_{ij}^*) can be any point in rectangle ij, which corresponds to the i^{th} subinterval of $[a, b]$ and the j^{th} subinterval of $[c, d]$.

$$\lim_{m,n\to\infty} \left(\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \right) = V$$

means that for every $\varepsilon > 0$ [output accuracy] there is a number N [input accuracy] such that whenever $m, n > N$, no matter how the points (x_{ij}^*, y_{ij}^*) are chosen, we have

$$\left| \left(\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A \right) - V \right| < \varepsilon.$$

If $\iint_R f(x,y)\,dA$ exists, we say that f is *integrable* over the rectangle R.

Method 2 leads to a technique for computing double integrals:

Theorem (Fubini's Theorem): If f is continuous on $R = [a, b] \times [c, d]$ then f is integrable on R and

$$\iint_R f(x,y)\,dA = \overbrace{\int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{c,d \text{ limits on } y}\,dx}^{a,b \text{ limits on } x} = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

Note: The textbook says this is proved in advanced mathematics courses. However, we can get a look at the ideas involved. We will post notes about this on the web page.

Example: Find the volume of the area above $R = [0,1] \times [0,2]$ and under the graph of $f(x,y) = xy$.

$$\int_0^1 \int_0^2 xy\,dy\,dx = \int_0^1 \left[\frac{xy^2}{2} \Bigg|_{y=0}^{y=2} \right] dx = \int_0^1 2x\,dx = x^2 \Big|_{x=0}^{x=1} = 1.$$

Notice that writing $\left[\frac{xy^2}{2} \Bigg|_{y=0}^{y=2} \right]$ rather than $\left[\frac{xy^2}{2} \Bigg|_{0}^{2} \right]$ helps to keep things straight.

Another way to approach this particular example: Note that in the inner integral $\int_0^2 xy\,dy$, we treat x as a constant. Therefore we can move it outside the integral sign:

$$\int_0^1 \int_0^2 xy\,dy\,dx = \int_0^1 x \int_0^2 y\,dy\,dx.$$

Now, the inner integral $\int_0^2 y\,dy$ actually is a constant, so we can move it outside the integral sign:

$$\int_0^1 x \int_0^2 y\,dy\,dx = \left[\int_0^1 x\,dx \right] \left[\int_0^2 y\,dy \right] = \left[\frac{x^2}{2} \Bigg|_{x=0}^{x=1} \right] \left[\frac{y^2}{2} \Bigg|_{y=0}^{y=2} \right] = \left(\frac{1}{2} \right) \left(\frac{4}{2} \right) = 1.$$

Proposition: If $f(x,y) = g(x)h(y)$ is a product of continuous functions of x and of y, and $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\,dA = \iint_R g(x)h(y)\,dA = \left(\int_a^b g(x)\,dx \right) \left(\int_c^d h(y)\,dy \right).$$

Another application: Average value.

The average value of $f(x,y)$ on R is given by

$$\frac{\iint_R f(x,y)\,dA}{\text{area(R)}}.$$

Example: Find the average value of the function $f(x,y) = y\cos(xy)$ on the rectangle $R = \left[0, \frac{\pi}{2}\right] \times [0,1]$.

$$\text{area(R)} = \left(\frac{\pi}{2} - 0\right)(1-0) = \frac{\pi}{2}.$$

$$\iint_R f(x,y)\,dA = \int_0^{\frac{\pi}{2}} \int_0^1 y\cos(xy)\,dy\,dx.$$

We can integrate $y\cos(xy)$ with respect to y using integration by parts. But it's easier to change the order of integration.

$$\iint_R f(x,y)\,dA = \int_0^1 \int_0^{\frac{\pi}{2}} y\cos(xy)\,dx\,dy = \int_0^1 \left[\sin(xy)\Big|_{x=0}^{x=\frac{\pi}{2}}\right] dy = \int_0^1 \sin\left(\frac{\pi}{2}y\right) dy =$$

$$\frac{2}{\pi}\left(-\cos\left(\frac{\pi}{2}y\right)\right)\Big|_{y=0}^{y=1} = \frac{2}{\pi}.$$

$$\text{Average value of } f \text{ on } R = \frac{\frac{2}{\pi}}{\frac{\pi}{2}} = \frac{4}{\pi^2}.$$

Example: Evaluate

$$\int_1^2 \int_0^1 xe^{xy}\,dx\,dy.$$

Example: Find the volume of the region below the plane $z = 1$, above the surface $z = x^2 - 5$, and between the planes $y = 0$ and $y = 1$.

Hint: First try to draw the region. Pay particular attention to the intersection of the top and bottom surfaces.

Notice that you are finding the volume of a region between two graphs. Think back to single-variable calculus and finding the area of a region between two graphs.

Example: Determine whether the average value of $f(x, y) = g(x)h(y)$ on a rectangle $R = [a, b] \times [c, d]$ is equal to the product of the average value of $g(x)$ on $[a, b]$ and the average value of $h(y)$ on $[c, d]$.

Note that the average value of $g(x)$ on $[a, b]$ is given by

$$\frac{\displaystyle\int_a^b g(x)\,dx}{\text{length([a, b])}}.$$

Example: Show that the volume below the surface $z = f(x, y)$ and above the rectangle R equals the product of the area of R and the average height of the surface on R.

Assuming that this principle also holds for regions R that are not rectangles, find the average value of $f(x, y) = \sqrt{1 - x^2 - y^2}$ on the unit disc $x^2 + y^2 \leq 1$.