Here are some solutions. As always no guarantee there are no typos, but I think things are correct.

1. Consider two vector fields $\mathbf{F} = \langle -y, x\rangle$ and $\mathbf{G} = \langle \cos x + y, x - 1\rangle$ defined in the plane.

   (a) Determine whether $\mathbf{F}$ or $\mathbf{G}$ is conservative. If conservative, produce a potential function.

   We note that $\mathbf{F}$ has nonzero curl, so $\mathbf{F}$ cannot be conservative. The curl of $\mathbf{G}$ is zero, and $\mathbf{G}$ is smooth, so we look for a potential function g with $\mathbf{G} = \nabla g$. We would need $g_x = \cos x + y$ and $g_y = x - 1$.

   $g_x = \cos x + y$ implies $g(x, y) = \sin x + xy + h(y)$, so that $g_y = x + h'(y) = x - 1$, thus $h'(y) = -1$ and we can take $h(y) = -y$. We double check that with $g(x, y) = \sin x + xy - y$, $\mathbf{G} = \nabla g$.

   (b) Let C be the oriented curve from $A = (-3, 0)$ to $B = (1, 0)$ given as follows: the straight line from $(-3, 0)$ to $(-1, 0)$, then the clockwise arc of the unit circle to the point $(1, 0)$. Compute the line integrals $\int_C \mathbf{F} \bullet d\mathbf{r}$ and $\int_C \mathbf{G} \bullet d\mathbf{r}$.

   $\int_C \mathbf{G} \bullet d\mathbf{r}$ is certainly easier since we can use the fundamental theorem of line integrals:

   $\int_C \mathbf{G} \bullet d\mathbf{r} = \int_C \nabla g \bullet d\mathbf{r} = g(B) - g(A) = g(1, 0) - g(-3, 0) = \sin 1 - \sin(-3)$.

   For the other line integral, we need to parametrize the two pieces of the curve. Let C_1 be the first piece given by $\mathbf{r}(t) = \langle t, 0\rangle$ for $-3 \le t \le -1$, and let C_2 be the second piece given by $\mathbf{r}(t) = \langle -\cos t, \sin t\rangle$ for $0 \le t \le \pi$. Note the $-\cos t$ to produce the clockwise orientation. So now $\int_C \mathbf{F} \bullet d\mathbf{r} = \int_{C_1} \mathbf{F} \bullet d\mathbf{r} + \int_{C_2} \mathbf{F} \bullet d\mathbf{r} = \int_{-3}^{-1} \langle 0, t\rangle \bullet \langle 1, 0\rangle dt + \int_0^{\pi} \langle -\sin t, -\cos t\rangle \bullet \langle \sin t, \cos t\rangle dt = 0 - \pi = -\pi$.

2. Let M be the surface of the potato chip which is that part of the surface $z = xy$ inside the cylinder $x^2 + y^2 = 1$, and let C be its boundary positively oriented. If $\mathbf{F} = \langle 3xz - y, xz + yz, x^2 + y^2\rangle$, find $\displaystyle\oint_C \mathbf{F} \bullet d\mathbf{r}$.

   By Stokes' theorem, $\displaystyle\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_M \nabla \times \mathbf{F} \bullet d\mathbf{S}$. The curl is $\nabla \times \mathbf{F} = \langle y - x, x, z + 1\rangle$. Parametrizing M as the graph of a function $f(x, y) = xy$ with the parametrization domain the unit disk $D = \{(x, y) \mid x^2 + y^2 \le 1\}$ we get $d\mathbf{S} = \langle -f_x, -f_y, 1\rangle dA$, and so $\displaystyle\oint_C \mathbf{F} \bullet d\mathbf{r} = \iint_M \langle y - x, x, z + 1\rangle \bullet d\mathbf{S} = \iint_D \langle y - x, x, xy + 1\rangle \bullet \langle -y, -x, 1\rangle dA = \iint_D (1 + 2xy - x^2 - y^2) dA$. Noting that $\iint_D 2xy dA = 0$ by symmetry and changing to polar coordinates for the rest, we obtain $\displaystyle\oint_C \mathbf{F} \bullet d\mathbf{r} = \int_0^{2\pi} \int_0^1 (1 - r^2) r\, dr\, d\theta = 2\pi(1/2 - 1/4) = \pi/2$.

3. Let E denote the portion of the solid sphere of radius R in the first octant, and let $\mathbf{F} = \langle 2x + y, y^2, \cos(xy)\rangle$. Compute the flux of $\mathbf{F}$ (surface integral) across the boundary of E, oriented by the outward-pointing normal vectors.

By the Divergence theorem, the flux across the boundary is equal to $\iiint_E \nabla \bullet \mathbf{F}\, dV = \iiint_E 2 + 2y\, dV = 2\iiint_E dV + 2\iiint_E y dV$, and we note that the first integral is just twice the volume of one-eighth of a sphere of radius R, $\pi R^3/3$. In spherical coordinates (note are going from xyz to $\rho\theta\phi$ so the Jacobian appears) we can express the second integral as $2\int_0^{\pi/2}\int_0^{\pi/2}\int_0^R \rho\sin\phi\sin\theta\ \rho^2\sin\phi\, d\rho\, d\theta\, d\phi = \dfrac{R^4}{2}\int_0^{\pi/2}\sin(\phi)^2\, d\phi = \dfrac{\pi R^4}{8}$, for a final answer of $\pi R^3/3 + \pi R^4/8$.

4. Let C denote the circle of radius R centered at the origin and oriented counterclockwise. Let $\mathbf{F} = \langle \arctan x + y^3, 2x - \sqrt[3]{y}\rangle$. Compute $\oint_C \mathbf{F} \bullet d\mathbf{r}$.

   We can use Green's theorem: Let D be the disk whose boundary is C. Then $\oint_C \mathbf{F}\bullet d\mathbf{r} =$
   $\iint_D (Q_x - P_y)\, dA = \iint_D (2 - 3y^2)\, dA = \int_0^{2\pi}\int_0^R (2 - 3r^2\sin^2\theta) r\, dr\, d\theta =$
   $\int_0^{2\pi}\int_0^R 2r\, dr\, d\theta - \int_0^{2\pi}\sin^2\theta\, d\theta \int_0^R 3r^3\, dr = 2\pi R^2 - 3\pi R^4/4.$

5. Compute the flux of the vector field $\mathbf{F} = \langle x^3, 2xz^2, 3y^2z\rangle$ over the surface M where M is the boundary of the solid E bounded by the paraboloid $z = 4 - x^2 - y^2$ and the xy-plane.

   Using the Divergence theorem and cylindrical coordinates, we get
   $\iint_M \mathbf{F}\bullet d\mathbf{r} = \iiint_E \nabla\bullet\mathbf{F}\, dV = \iiint_E (3x^2 + 3y^2)\, dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r^2\, r\, dz\, dr\, d\theta =$
   $2\pi\int_0^2 3r^3(4 - r^2)\, dr = 32\pi.$

6. Compute $\int_C y\, dx + x\, dy + (x^2 + y^2)\, dz$ where C is the positively oriented curve which bounds that part of the unit sphere in the first octant. Note that this is a closed curve consisting of three parts. Let M denote the corresponding surface.

   By Stokes' theorem, $\oint_C \mathbf{F}\bullet d\mathbf{r} = \iint_M \nabla\times\mathbf{F}\bullet d\mathbf{S}$. Now $\nabla\times\mathbf{F} = \langle 2y, -2x, 0\rangle$. Also given that the surface is a level surface: $G(x, y, z) = x^2 + y^2 + z^2 = 1$, a normal vector is $\nabla G = \langle 2x, 2y, 2z\rangle$. Since this is the unit sphere, the unit normal $\mathbf{n} = \langle x, y, z\rangle$ is the outward facing unit normal vector, so $d\mathbf{S} = \mathbf{n} dS$. Thus $\iint_M \nabla\times\mathbf{F}\bullet d\mathbf{S} = \iint_M \langle 2y, -2x, 0\rangle \bullet \langle x, y, z\rangle dS = 0$.