Math 11 Exam, Fall 2005, Solutions

Problem 1

(a) The direction in which the temperature will rise most rapidly is given by the gradient, so the direction in which the temperature will fall most rapidly is given by the negative of the gradient. The temperature gradient is

$$\nabla T(x,y,z) = 1000e^{x^2-4y^2-9z^2}\langle 2x,-8y,-18z\rangle$$

So the temperature gradient at the position of the ship is

$$\nabla T(2,1,0) = 1000\langle 4,-8,0\rangle$$

and its magnitude is

$$|\nabla T(2,1,0)| = 1000\sqrt{4^2+8^2} = 1000\sqrt{80} = 4000\sqrt{5}$$

Hence the direction the ship should proceed is given by the unit vector

$$\begin{aligned} -\frac{\nabla T(2,1,0)}{|\nabla T(2,1,0)|} &= \frac{-1}{4000\sqrt{5}}\langle 4000,-8000,0\rangle \\ &= \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\rangle \end{aligned}$$

(b) The rate at which the ship will cool in a given direction is given by the directional derivative in that direction. In this case, they want to move in the direction

$$\mathbf{u} = -\frac{\nabla T(2,1,0)}{|\nabla T(2,1,0)|} = \langle \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\rangle$$

Since the angle between $\mathbf{u}$ and $\nabla T(2,1,0)$ is $\theta = \pi$,

$$\begin{aligned} D_{\mathbf{u}}T(2,1,0) &= \nabla T(2,1,0)\cdot\mathbf{u} \\ &= |\nabla T(2,1,0)||\mathbf{u}|\cos(\pi) \\ &= -|\nabla T(2,1,0)| \\ &= -4000\sqrt{5} \end{aligned}$$

So the ship will cool at a rate of $-4000\sqrt{5}$ degrees per unit distance (the negative sign indicates that the temperature is decreasing).

(c) First, note that the direction of maximum increase of T is given by the temperature gradient ∇T. Again, the rate that the ship will cool off (or heat up) in a direction specified by a unit vector $\mathbf{u}$ is given by the directional derivative, $D_{\mathbf{u}}T(2,1,0) = \nabla T(2,1,0)\cdot\mathbf{u} = |\nabla T(2,1,0)|\cos(\theta)$, where θ is the angle between $\nabla T(2,1,0)$ and $\mathbf{u}$. So in this case,

$$-500\sqrt{80} \le D_{\mathbf{u}}T(2,1,0) = |\nabla T(2,1,0)|\cos(\theta) = 4000\sqrt{5}\cos(\theta)$$

Hence

$$\cos(\theta) \geq \frac{-500\sqrt{80}}{4000\sqrt{5}} = \frac{-\sqrt{5}}{2\sqrt{5}} = \frac{-1}{2}$$

Since the angle between any two vector lies in the range $0 \leq \theta \leq \pi$,

$$0 \leq \theta \leq \frac{2\pi}{3}$$

Problem 2

(a) The model plane will make contact with the hill when its coordinates satisfy the equations defining the surface of the hill. So

$$\begin{aligned} 200 - 4t &= 100 - (2\sin(t))^2 - (2\cos(t))^2 \\ 100 - 4t &= -4 \\ t &= 26 \end{aligned}$$

(b) The crash will occur at

$$\mathbf{r}(26) = \langle 2\sin(26), 2\cos(26), 96\rangle$$

(c) The length of the flight path is given by the arc length of $\mathbf{r}$, $0 \leq t \leq 26$. Since

$$|\mathbf{r}'(t)| = |\langle 2\cos(t), -2\sin(t), -4\rangle| = \sqrt{4+16} = 2\sqrt{5}$$

Hence the length of the flight path is given by

$$\int_0^{26} |\mathbf{r}'(t)| dt = \int_0^{26} 2\sqrt{5}\ dt = 26 \cdot 2\sqrt{5} = 52\sqrt{2}$$

(d) The angle at which the plane hits the ground is given by the angle between the tangent vector to $\mathbf{r}$ at $t = 50$ and the downward normal to the ground.

$$\begin{aligned} \mathbf{r}'(50) &= \langle 2\cos(50), -2\sin(50), -4\rangle \\ \mathbf{n} &= \langle 0, 0, -1\rangle \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{r}'(50) \cdot \mathbf{n} &= |\mathbf{r}'(50)||\mathbf{n}|\cos(\theta) \\ 4 &= 2\sqrt{5}\cos(\theta) \\ \theta &= \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) \end{aligned}$$

Problem 3

Consider the tangent plane T to the surface $S = x^2 + y^2 + z^2 = 9$ at the point $(1, 2, 2)$. Find the point of intersection of T and the x-axis.

ANSWER: We begin by finding an expression for T. The surface S is a level surface for the function $F(x, y, z) = x^2 + y^2 + z^2$, so $\nabla F(1, 2, 2)$, being normal to the level surface at the point $(1, 2, 2)$, will be normal to the tangent plane T.

We calculate:

$$\nabla F(1,2,2) = \langle \frac{\partial F}{\partial x}(1,2,2), \frac{\partial F}{\partial y}(1,2,2), \frac{\partial F}{\partial z}(1,2,2) \rangle = \langle 2, 4, 4 \rangle.$$

Thus, an equation for T is

$$0 = 2(x-1) + 4(y-2) + 4(z-2) \Leftrightarrow 2x + 4y + 4z = 18.$$

This plane intersects the x-axis when $y = z = 0$; in other words, when $x = 9$.

Problem 4

Suppose that $F(x, y)$ is differentiable on all of $\mathbb{R}^2$ and let

$$g(s,t) = F(s^2 - t^2, t^2 - s^2).$$

Show that g satisfies the partial differential equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

ANSWER: Let $u(s, t) = s^2 - t^2, v(s, t) = t^2 - s^2$. Then we can write $g(s, t) = F(u, v)$.

By the chain rule,

$$\frac{\partial g}{\partial s} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial s} = 2s\frac{\partial F}{\partial u} - 2s\frac{\partial F}{\partial v}$$

and similarly

$$\frac{\partial g}{\partial t} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t} = -2t\frac{\partial F}{\partial u} + 2t\frac{\partial F}{\partial v}.$$

Thus,

$$t\frac{\partial g}{\partial s} = 2ts\left(\frac{\partial F}{\partial u} - \frac{\partial F}{\partial v}\right)$$

and

$$s\frac{\partial g}{\partial t} = 2st\left(\frac{\partial F}{\partial v} - \frac{\partial F}{\partial v}\right).$$

It follows that

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

as desired.

Solutions #5 · Multiple Choice

a) D $\qquad \int_a^b |\vec{r}'(t)|\,dt$ = arc length

b) E $\qquad$ the limit does NOT exist.

let $x=0$ and $f(0,y) = \frac{\sin(0)}{y^2} = \frac{0}{y^2} = 0$

let $x=y$ and $f(x,x) = \frac{\sin(x^2)}{2x^2}$, using l'hopital we have

$$\lim_{x\to 0} \frac{\sin(x^2)}{2x^2} = \lim_{x\to 0} \frac{2x\cdot\cos(x^2)}{4x} = \lim_{x\to 0} \frac{1}{2}\cdot\cos(x^2) = \frac{1}{2}$$

c) B $\qquad$ the limit of a continuous function is always defined and equals the limit as $(x,y)\to(a,b)$ along <u>ANY</u> path.

d) E $\qquad$ use the Fundamental Theorem of Calculus:

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\int_x^y \cos(t^3)\,dt = -\frac{d}{dx}\int_y^x \cos(t^3)\,dt = -\cos(x^3)$$

e) B note that $\vec{u} + \vec{v} = \langle 1, 0 \rangle = \hat{i}$ so that

$$\frac{\partial f}{\partial x} = D_{\hat{i}} f = D_{\hat{u}+\hat{v}} f = \nabla f \cdot (\hat{u} + \hat{v}) = \nabla f \cdot \vec{u} + \nabla f \cdot \vec{v}$$

$$= D_{\vec{u}} f + D_{\vec{v}} f = 2 - 1 = 1$$

f) D we know $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$

but $\frac{\partial y}{\partial t} = 0$ so that $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$

g) C ~~let~~ $\cancel{\vec{u} = \frac{1}{2} \langle -1, 2, -1 \rangle}$ ~~the~~

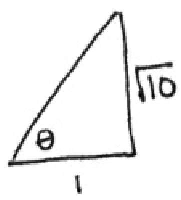
Let $\vec{u} = \frac{1}{\sqrt{6}} \cdot \langle -1, 2, -1 \rangle$ be the unit vector in direction $\langle -1, 2, -1 \rangle$

Then $D_{\vec{u}} f(7,8,9) = \nabla f(7,8,9) \cdot \vec{u} = \langle 3, 5, 7 \rangle \cdot \frac{1}{\sqrt{6}} \langle -1, 2, -1 \rangle$

$$= \frac{1}{\sqrt{6}} (-3 + 10 - 7) = 0$$

h) C We know $|\nabla f| = \sqrt{10}$ is the rate of change of the value of the function in the direction of the gradient.

Thus



so that $\theta = \arctan(\sqrt{10})$