1. The two curves $\mathbf{r}_1(t) = \langle 2, t, t^2 - 4\rangle$ and $\mathbf{r}_2(s) = \langle s, 3, 9 - s^2\rangle$ both lie on a surface S and intersect at some point P.

   (a) Find their point of intersection.

   (b) Find the angle between the tangent vectors of the curves at the point of intersection.

   (c) Find the equation of the tangent plane to S at the given point.

   Solution. *If the intersect, we must have from the* x, y *coordinates, that* $s = 2$ *and* $t = 3$. *Those choices give consistent* z *values: the intersect at* $(2, 3, 5)$

   We compute the velocity vectors $\mathbf{r}_1'(t) = \langle 0, 1, 2t\rangle$ *and* $\mathbf{r}_2'(s) = \langle 1, 0, -2s\rangle$. *The tangent vectors at the point are* $\mathbf{r}_1'(3) = \langle 0, 1, 6\rangle$, *and* $\mathbf{r}_2'(2) = \langle 1, 0, -4\rangle$. *We compute their cross product:*

$$\mathbf{r}_1'(3) \times \mathbf{r}_2'(2) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 6 \\ 1 & 0 & -4 \end{vmatrix} = \langle -4, 6, -1\rangle.$$

   Then the plane has equation $-4(x - 2) + 6(y - 3) - (z - 5) = 0$.

2. Does the limit

$$\lim_{(x,y)\to(1,0)} \frac{xy - y}{(x-1)^2 + y^2}$$

   exist? If so, what is its value?

   Solution. *The limit does not exist. If we approach* $(1, 0)$ *along the line* $x = 1$, *the limit is*

$$\lim_{y\to 0} \frac{y - y}{y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0.$$

   However, if we approach along the line $x = y + 1$, *then the limit is*

$$\lim_{y\to 0} \frac{y^2 + y - y}{y^2 + y^2} = \lim_{y\to 0} \frac{1}{2} \neq 0.$$

3. Consider two helixes, one parametrized as $\mathbf{r}_1(t) = \langle \cos t, \sin t, t\rangle$ and the other as $\mathbf{r}_2(s) = \langle \cos s, \sin s, s^2\rangle$. Particles move along the curves both starting out from $(1, 0, 0)$ at $s = t = 0$.

   (a) The particle on which curve starts out faster and why?

   (b) Eventually the other particle travels faster, but there is a moment when each particle has traveled exactly the same distance. Write an expression which — if solved — would reveal the time. Do not attempt to solve for the time.

Solution. *We have that* $\mathbf{r}_1'(t) = \langle -\sin t, \cos t, 1\rangle$ *and* $\mathbf{r}_2'(s) = \langle -\sin s, \cos s, 2s\rangle$ *with corresponding speeds* $\sqrt{2}$ *versus* $\sqrt{1+4s^2}$*. So at the start, the particle on the first curve is moving at* $\sqrt{2}$ *units/sec, while the particle on the second curve is only moving at 1 unit/sec.*

Clearly for $s > 1/2$*, the particle on the second curve starts to move faster.*

To find the time T *where the particles have traveled the same distance, we would have to solve the equation*

$$\int_0^T \sqrt{1+4s^2}\,ds = \int_0^T \sqrt{2}\,dt = \sqrt{2}T,$$

but while the integral can be done, solving it is off task.

4. The functions $\mathbf{r}_1(t) = \langle \cos t, \sin t, t\rangle$ and $\mathbf{r}_2(t) = \langle 1+t, t^2, t^3\rangle$ are parametrizations of two curves that intersect when $t = 0$.

   (a) Determine the angle of intersection between the two curves at the point of intersection.

   (b) Assuming that particles on these curves are moving with the same parameter t, is there another point in time (positive or negative) where these curves will intersect? Hint: it would help to think about the curves geometrically before making any algebraic argument.

Solution. *We see that* $\mathbf{r}_1(0) = \mathbf{r}_2(0) = \langle 1, 0, 0\rangle$*. To compute the angle between the curves, we compute the angle between their velocity vectors,* $\mathbf{r}_1'(t) = \langle -\sin t, \cos t, 1\rangle$*, and* $\mathbf{r}_2'(t) = \langle 1, 2t, 3t^2\rangle$*, so* $\mathbf{r}_1'(0) = \langle 0, 1, 1\rangle$*, and* $\mathbf{r}_2(0) = \langle 1, 0, 0\rangle$*. The dot product* $\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0) = 0$*, so they are orthogonal.*

As to the second part, we see that the first curve is a helix whose x *coordinate lies in* $[-1, 1]$*. We see that for* $t > 0$ *the* x *coordinate of the second curve is greater than one, so there is no intersection for* $t > 0$*. For negative* t*, the only chance for intersection is when* $t \in [-2, 0)$ *since that gives a viable* x *coordinate. But we observe on that interval,* $\sin t < 0$ *while* $t^2 > 0$*, so there is no further point of intersection.*

5. Find the points on the hyperboloid $x^2 + 4y^2 - z^2 = 4$ where the tangent plane is parallel to $2x + 2y + z = 5$.

Solution. *The hyperboloid is a level surface of the function* $F(x, y, z) = x^2 + 4y^2 - z^2$*, so a normal vector to the surface is given by the gradient* $\nabla F = \langle 2x, 8y, -2z \rangle$*. The tangent plane will be parallel to the given plane iff the normal vector to the surface is parallel to the normal vector of the plane:* $\mathbf{n} = \langle 2, 2, 1 \rangle$*. So we have* $\langle 2x, 8y, -2z \rangle = \lambda \langle 2, 2, 1 \rangle$*, so* $x = \lambda$*,* $y = \lambda/4$*, and* $z = -\lambda/2$*. To lie on the surface we must have* $\lambda^2 + 4(\lambda/4)^2 - (\lambda/2)^2 = 4 = \lambda^2$*. So* $\lambda = \pm 2$ *and the points in question are* $(2, 1/2, -1)$ *and* $(-2, -1/2, 1)$*.*

6. (30) [Multiple choice] (No partial credit) **Circle the correct answer.**

(a) A bug is crawling straight up the side of a cylinder of radius 2 feet whose equation is $x^2 + y^2 = 4$. The cylinder is rotating counterclockwise at a rate of one rotation per second. If the bug's total speed is 5π feet per second, which vector equation describes the bug's path through space?

A. $\langle 2\pi \cos(t), 2\pi \sin(t), 3\pi t \rangle$ **B**. $\langle 2\pi \cos(t), 2\pi \sin(t), 5\pi t \rangle$

C. $\langle 2 \cos(2\pi t), 2 \sin(2\pi t), 3\pi t \rangle$ **D**. $\langle 2 \cos(2\pi t), 2 \sin(2\pi t), 5\pi t \rangle$

E. $\langle 3 \cos(2\pi t), 3 \sin(2\pi t), 3\pi t \rangle$

Solution. *C. First parameterize the cylinder, then solve the equation*

$$\textit{speed}\langle 2\cos(2\pi t), 2\sin(2\pi t), xt \rangle = 5\pi$$

for x*.*