Your name: Danny Wong

Instructor (please circle): Barnett Van Erp

Math 11 Fall 2010: written part of HW9 (due Mon Nov 29)

Please show your work. No credit is given for solutions without justification.

(1) [8 points]

(a) Let $\mathbf{F} = (2xy,\ x^2 + 2yz,\ y^2)$ be a vector field in $\mathbb{R}^3$. Is there a scalar field f such that $\nabla f = \mathbf{F}$? Explain.

$\vec{P}$ $\vec{Q}$ $\vec{R}$

$$\text{curl } \vec{F} = \left(\frac{\partial \vec{R}}{\partial y} - \frac{\partial \vec{Q}}{\partial z}\right)\vec{i} + \left(\frac{\partial \vec{P}}{\partial z} - \frac{\partial \vec{R}}{\partial x}\right)\vec{j} + \left(\frac{\partial \vec{Q}}{\partial x} - \frac{\partial \vec{P}}{\partial y}\right)\vec{k}$$

$$= (2y-2y)\vec{i} + (0-0)\vec{j} + (2x-2x)\vec{k}$$

$$= \vec{0}$$

$\boxed{\text{YES}}$

curl $\vec{F} = \vec{0} \rightarrow \vec{F}$ is a conservative vector field

that is, there exists a function f such that $\vec{\nabla} f = \vec{F}$.

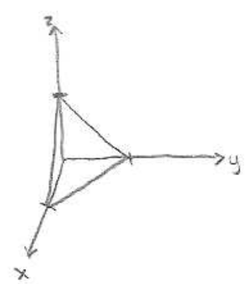
(b) Is there a vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \mathbf{F}$? Explain.

$$\text{div } \vec{F} = \frac{\partial \vec{P}}{\partial x} + \frac{\partial \vec{Q}}{\partial y} + \frac{\partial \vec{R}}{\partial z} = 2y + 2z + 0 = 2y + 2z$$

$\boxed{\text{NO}}$

- if it were true that $\vec{F} = \text{curl } \vec{G}$, then div $\vec{F}$ = div curl $\vec{G} = 0$
- however, div $\vec{F} \neq 0$
- thus, there is not a vector field $\vec{G}$ such that curl $\vec{G} = \vec{F}$

(c) Let C be the triangle formed by the boundary of the plane $x + y + z = 1$ restricted to the first octant, traversed in a counter-clockwise sense when viewed in the xy-plane. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$. If you make use of one of your above answers (and we suggest you do), explain how.



- line integrals of conservative vector fields are independent of path
- there is a theorem that states $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in D
- since $\vec{F} = (2xy, x^2+2yz, y^2)$ is a conservative vector field, and the triangle formed by the boundary of the plane restricted to the first octant is a closed path, $\int_C \vec{F} \cdot d\vec{r} = 0$

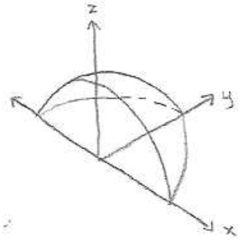
$\boxed{0}$

(2) [8 points] Let S be the part of the sphere of radius 2 centered at the origin, lying in the region $y \geq 0$, $z \geq 0$. Compute $\iint\limits_S yz\, dS$

$x = 2\sin\phi\cos\theta$

$y = 2\sin\phi\sin\theta$

$z = 2\cos\phi$



$0 \leq \phi \leq \pi/2 \qquad u = \phi$

$0 \leq \theta \leq \pi \qquad v = \theta$

$\vec{r}_u = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi \rangle$

$\vec{r}_v = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0 \rangle$

$$\vec{r}_u \times \vec{r}_v = \langle (2\cos\phi\sin\theta)(0) - (-2\sin\phi)(2\sin\phi\cos\theta), (-2\sin\phi)(-2\sin\phi\sin\theta) - (2\cos\phi\cos\theta)(0), (2\cos\phi\cos\theta)(2\sin\phi\cos\theta) - (2\cos\phi\sin\theta)(-2\sin\phi\sin\theta) \rangle$$

$$= \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, \underbrace{4\sin\phi\cos\phi\cos^2\theta + 4\sin\phi\cos\phi\sin^2\theta}_{4\sin\phi\cos\phi} \rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{16\sin^4\phi\cos^2\theta + 16\sin^4\phi\sin^2\theta + 16\sin^2\phi\cos^2\phi}$$

$$= 4\sin\phi\sqrt{\underbrace{\underbrace{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta}_{\sin^2\phi} + \cos^2\phi}_{1}}$$

$$= 4\sin\phi$$

$$\int_0^{\pi}\int_0^{\pi/2} (2\sin\phi\sin\theta)(2\cos\phi)(4\sin\phi)\, d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{\pi/2} 16\sin^2\phi\cos\phi\sin\theta\, d\phi\, d\theta$$

$$= 16 \cdot \int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi \cdot \int_0^{\pi} \sin\theta\, d\theta$$

$u = \sin\phi$

$du = \cos\phi\, d\phi$

$$= 16 \cdot \int_0^1 u^2\, du \cdot \int_0^{\pi} \sin\theta\, d\theta$$

$$= 16 \cdot \left(\frac{u^3}{3} \Big|_0^1 \right) \cdot \left(-\cos\theta \Big|_0^{\pi} \right)$$

$$= 16 \cdot \frac{1}{3} \cdot 2$$

$$= \boxed{32/3}$$

(3) [10 points] Let S be the part of the paraboloid $z = x^2 + y^2$ with $z \leq 1$, with surface normal oriented upwards. Let $\mathbf{F}$ be the vector field $(xz,\ yz,\ 1)$

(a) Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

$z = g(x,y) = x^2+y^2$ $\quad (\vec{r}_u \times \vec{r}_v) = \left(-\frac{\partial g}{\partial x}\vec{i} - \frac{\partial g}{\partial y}\vec{j} + \vec{k}\right) = \left(-2x\vec{i} - 2y\vec{j} + \vec{k}\right)$

$u = x$

$v = y$ $\quad \vec{F}\cdot(\vec{r}_u \times \vec{r}_v) = \langle xz, yz, 1\rangle \cdot \langle -2x, -2y, 1\rangle = -2x^2z - 2y^2z + 1$

$= -2x^2(x^2+y^2) - 2y^2(x^2+y^2) + 1$

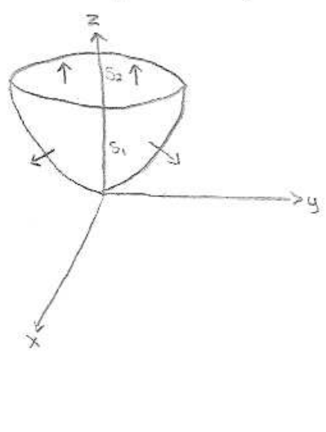
$= -2x^4 - 2x^2y^2 - 2x^2y^2 - 2y^4 + 1$

$= -2x^4 - 4x^2y^2 - 2y^4 + 1$

$= 1 - 2(x^2+y^2)^2$

$$\iint_D 1-2(x^2+y^2)^2\,dA = \int_0^{2\pi}\int_0^1 (1-2r^4)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r-2r^5\,dr\,d\theta = 2\pi\left(\frac{r^2}{2} - \frac{r^6}{3}\Big|_0^1\right) = 2\pi\left(\frac{1}{2}-\frac{1}{3}\right) = 2\pi\left(\frac{1}{6}\right) = \boxed{\frac{\pi}{3}}$$

(b) Use this to evaluate $\iint_T \mathbf{F} \cdot d\mathbf{S}$ where T is the *entire* surface of the solid truncated paraboloid bounded by $z = x^2 + y^2$ and $z = 1$, with surface normal everywhere oriented *outwards*:



$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_{S_2} \vec{F}\cdot\vec{k}\,dS = \iint_D 1\,dA = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = 2\pi\left(\frac{r^2}{2}\Big|_0^1\right) = \pi$$

$$\iint_{S_1} \vec{F}\cdot(-\vec{n})\,dS = -\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = -\frac{\pi}{3}$$ (negative of the answer to part a)

$$\pi - \frac{\pi}{3} = \boxed{\frac{2\pi}{3}}$$