Your name: Solutions

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Math 11 Fall 2010: written part of HW7 (due Wed Nov 10)
Please show your work. No credit is given for solutions without justification.

(1) [8 points]

(a) Find $\iiint_E f\,dV$ where $f(x,y,z) = \sin z$ and E is the solid region lying in the first octant bounded by $z = 1 - x^2 - y^2$.

let $r^2 = x^2 + y^2$

$$0 \le z \le 1 - r^2$$
$$\left.\begin{array}{l} 0 \le y \le \sqrt{1-x^2} \\ 0 \le x \le 1 \end{array}\right\} \left.\begin{array}{l} 0 \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 1 \end{array}\right\rangle \text{Quarter disk in xy plane.}$$

$$\int_0^{\frac{\pi}{2}}\int_0^1\int_0^{1-r^2} (\sin z)\, r\,dz\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 r\left(\int_0^{1-r^2}(\sin z)\,dz\right)dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r\left[-\cos z\right]_{z=0}^{z=1-r^2} dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^1 \left(r - r\cos(1-r^2)\right) dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2} + \frac{\sin(1-r^2)}{2}\right]_{r=0}^{r=1} d\theta = \int_0^{\frac{\pi}{2}}\left(\frac{1}{2} - \frac{1}{2}\sin 1\right)d\theta$$

$$= \frac{\pi}{4}(1 - \sin 1)$$

(b) Find the *average value* of this function f over the solid region E

$$\iiint_E dV = \int_0^{\frac{\pi}{2}}\int_0^1\int_0^{1-r^2} r\,dz\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 r(1-r^2)\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{4}\,d\theta$$

$$= \frac{\pi}{8}$$

$$\frac{\frac{\pi}{4}(1-\sin 1)}{\frac{\pi}{8}} = 2(1-\sin 1)$$

(2) [10 points] By converting to spherical coordinates, evaluate

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} y\,dz\,dx\,dy$$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

x, y are bounded by unit disk in xy plane

$\therefore\ 0 \le \theta \le 2\pi$

ϕ is bounded by z axis and $z = \sqrt{x^2+y^2}$

$\therefore\ 0 \le \phi \le \frac{\pi}{4}$

ρ is bound by $z = \sqrt{2-x^2-y^2}$, which is a sphere of radius $\sqrt{2}$.

$\therefore\ 0 \le \rho \le \sqrt{2}$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} \rho \sin\phi \sin\theta \left(\rho^2 \sin\phi\right) d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} \sin\theta \left(\int_0^{\frac{\pi}{4}} \sin^2\phi \left(\int_0^{\sqrt{2}} \rho^3\, d\rho \right) d\phi \right) d\theta$$

Integral of sine over its period is 0.
so this integral equals 0.

$= 0.$

(3) [10 points] Let C be the union of the straight line starting at $(0,0)$ and ending at $(2,1)$ with the quarter circle from $(2,1)$ to $(3,0)$ with center $(2,0)$ traversed clockwise.

(a) Compute $\int_C xy\,ds$

We parametrize C as follows:

$$x(t) = \begin{cases} 2t, & 0 \le t \le 1 \\ 2-\cos\left(\frac{\pi}{2}t\right), & 1 \le t \le 2 \end{cases}$$

$$y(t) = \begin{cases} t, & 0 \le t \le 1 \\ \sin\left(\frac{\pi}{2}t\right), & 1 \le t \le 2 \end{cases}$$

$$\int_0^2 x(t)y(t)\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^1 x(t)y(t)\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt + \int_1^2 x(t)y(t)\sqrt{\left(\tfrac{dx}{dt}\right)^2+\left(\tfrac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^1 2t^2\sqrt{2^2+1^2}\,dt + \int_1^2 \left(2-\cos\left(\tfrac{\pi}{2}t\right)\right)\sin\left(\tfrac{\pi}{2}t\right)\sqrt{\left(\tfrac{\pi}{2}\sin\left(\tfrac{\pi}{2}t\right)\right)^2+\left(\tfrac{\pi}{2}\cos\left(\tfrac{\pi}{2}t\right)\right)^2}\,dt$$

$$= \int_0^1 (2\sqrt{5})t^2\,dt + \int_1^2 \left(2-\cos\left(\tfrac{\pi}{2}t\right)\right)\left(\tfrac{\pi}{2}\sin\left(\tfrac{\pi}{2}t\right)\right)dt$$

$$= (2\sqrt{5})\tfrac{1}{3}t^3\Big]_0^1 + \tfrac{1}{2}\left(2-\cos\left(\tfrac{\pi}{2}t\right)\right)^2\Big]_1^2 = \frac{2\sqrt{5}}{3} - 0 + \frac{9}{2} - 2$$

$$= \frac{4\sqrt{5}+15}{6}$$

(b) Compute $\int_C y\,dx - x\,dy$

$$\int_C \langle y,-x\rangle \cdot d\vec{r} = \int_0^2 \langle y(t),-x(t)\rangle \cdot \left\langle \tfrac{dx}{dt}, \tfrac{dy}{dt}\right\rangle dt = \int_0^2 \left(y(t)\tfrac{dx}{dt} - x(t)\tfrac{dy}{dt}\right)dt$$

$$= \int_0^1 \left(y(t)\tfrac{dx}{dt} - x(t)\tfrac{dy}{dt}\right)dt + \int_1^2 \left(y(t)\tfrac{dx}{dt} - x(t)\tfrac{dy}{dt}\right)dt$$

$$= \int_0^1 \left(t(2) - 2t(1)\right)dt + \int_1^2 \left(\sin\left(\tfrac{\pi}{2}t\right)\left(\tfrac{\pi}{2}\sin\left(\tfrac{\pi}{2}t\right)\right) - \left(2-\cos\left(\tfrac{\pi}{2}t\right)\right)\left(\tfrac{\pi}{2}\cos\left(\tfrac{\pi}{2}t\right)\right)\right)dt$$

$$= \int_0^1 0\,dt + \int_1^2 \left(\tfrac{\pi}{2}\left(\sin^2\left(\tfrac{\pi}{2}t\right)+\cos^2\left(\tfrac{\pi}{2}t\right)\right) - \pi\cos\left(\tfrac{\pi}{2}t\right)\right)dt$$

$$= \int_1^2 \left(\tfrac{\pi}{2} - \pi\cos\left(\tfrac{\pi}{2}t\right)\right)dt = \tfrac{\pi}{2}t\Big]_1^2 - 2\sin\left(\tfrac{\pi}{2}t\right)\Big]_1^2$$

$$= \frac{\pi}{2} - 0 + 2$$

$$= \frac{\pi}{2} + 2.$$

(c) Describe how your answer to (a) and your answer to (b) would change if C were replaced with $-C$, that is, the same path traversed in the opposite sense.

In terms of the integrals, $\vec{r}(t)$ would be reversed in direction, which would make $\frac{dx}{dt}$ and $\frac{dy}{dt}$ change in sign.

The integral in part a), $\int_0^2 x(t)y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ would be unaffected.

However in part b), $\int_0^2 \langle y(t),-x(t)\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt$ would change in sign.