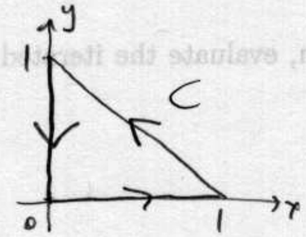
11/12/10
Barnett

MATH 11 WORKSHEET : Green's Theorem

A) 

curve C is triangle

Use Green's Thm. to evaluate $\int_C \vec{F} \cdot d\vec{r}$

for $\vec{F}(x,y) = \left(-xy^2 + 1,\ y^3\right)$

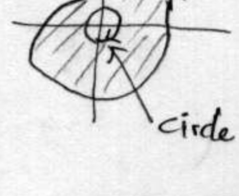
[Hint: turn it into a double integral]

B) Create a vector field $\vec{F}$ whose line int. $\int_C \vec{F} \cdot d\vec{r}$ on any closed curve is always the <u>area</u> enclosed by C:

C) Use Green's Thm to prove that if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in $\mathbb{R}^2$, line integrals of $\vec{F}$ are path-independent:

D) Use Green's Thm to explain why $\int_C \vec{F} \cdot d\vec{r} = 2\pi$ for <u>any</u> curve C that circles origin once, for $\vec{F} = \left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2}\right)$.

[Hint: consider D shown].



circle radius r.

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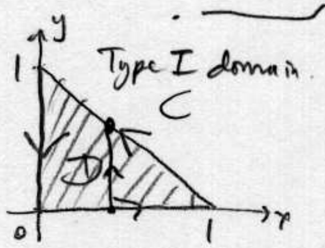
SOLUTIONS

A) 

curve C is triangle

Use Green's Thm. to evaluate $\int_C \vec{F}\cdot d\vec{r}$

for $\vec{F}(x,y) = (\overbrace{-xy^2+1}^{P},\ \overbrace{y^3}^{Q})$

[Hint: turn it into a double integral]

$$\int_C \vec{F}\cdot d\vec{r} \overset{G.T.}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \left(0 - (-2xy)\right) dA$$

$$= \int_0^1 \int_0^{1-x} 2xy \, dy\, dx = \int_0^1 x \cdot \underbrace{\left[y^2\right]_{y=0}^{y=1-x}}_{(1-x)^2} dx = \int_0^1 (x - 2x^2 + x^3)\, dx$$

$$= \tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} = 1/12$$

B) Create a vector field $\vec{F}$ whose line int. $\int_C \vec{F}\cdot d\vec{r}$ on any closed curve is always the area enclosed by C: $\vec{F} = (-y, 0)$ or $(0, x)$ or $(-y/2, x/2)$

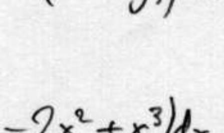
all have $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = +1$, so G.T. gives L.I. as area.

C) Use Green's Thm to prove that if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ in $\mathbb{R}^2$, line integrals of $\vec{F}$ are path-independent:

For any simple closed curve C,

$$0 = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \overset{G.T.}{=} \int_C \vec{F}\cdot d\vec{r}$$

D is interior of C

so 

all closed path integrals vanish.

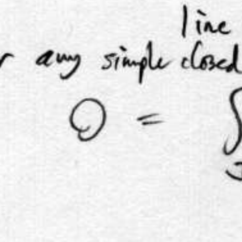
Given points A & B

$C = C_1 - C_2$ is closed path, so $\int_{C_1 - C_2} \vec{F}\cdot d\vec{r} = 0$.

so $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$

D) Use Green's Thm to explain why $\int_C \vec{F}\cdot d\vec{r} = 2\pi$ for any curve C that circles origin once, for $\vec{F} = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.

[Hint: consider D shown].



$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$ everywhere in D shown,

so $\int_{C_1} \vec{F}\cdot d\vec{r} = \underbrace{-\int_{C_2} \vec{F}\cdot d\vec{r}}_{\text{since L.I. around CW circle is } -2\pi} = +2\pi$ for any C_1