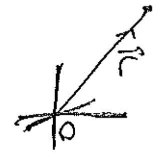
CALCULUS III Midterm 2 SOLUTIONS. M Apr 7. 2003. NYU.

1) $f(x,y,z) = $ pythagorean distance $= \sqrt{x^2+y^2+z^2}$



a) $\frac{\partial f}{\partial x} = 2x \cdot \frac{1}{2}(x^2+y^2+z^2)^{-1/2}$ using chain rule (single variable)

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} = g(x,y,z)$$

b) try approach origin along x-axis: $\lim_{x\to 0,\, y=0,\, z=0} g(x,y,z) = \frac{x}{\sqrt{x^2+0^2+0^2}} = \frac{x}{\sqrt{x^2}} = +1$

or -1 if $x<0$, since $\frac{x}{\sqrt{x^2}} = \text{sign}(x)$.

I removed up to 1 point for incorrect statements.

y-axis $\lim_{x=0,\, y\to 0,\, z=0} g(x,y,z) = \frac{0}{\sqrt{0^2+y^2+0^2}} = 0.$

Not continuous since limit of approaching origin along two paths are not the same. You also can see $g(0,0,0)$ = undefined, so g cannot be continuous at that point (not in domain of g).

NB. this does not prove that lim doesn't exist!

2) $f_x = \frac{2x}{3} \cdot \left(\frac{x^2+y}{3}\right)^{-1} = \frac{2x}{x^2+y} \xrightarrow{(x_0,y_0)=(1,2)} \frac{2}{1+2} = \frac{2}{3}$

$f_y = \frac{1}{3} \cdot \left(\frac{x^2+y}{3}\right)^{-1} = \frac{1}{x^2+y} \xrightarrow{(x_0,y_0)=(1,2)} \frac{1}{1+2} = \frac{1}{3}$

a) About (x_0,y_0), linear approx is

$$L(x,y) = f(x_0,y_0) + f_x\Big|_{(x_0,y_0)}(x-x_0) + f_y\Big|_{(x_0,y_0)}(y-y_0)$$

$$= 1 + \underbrace{\ln\left(\frac{1+2}{3}\right)}_{0} + \frac{2}{3}(x-1) + \frac{1}{3}(y-2)$$

Or, could write (although not needed), $L(x,y) = -\frac{1}{3} + \frac{2}{3}x + \frac{1}{3}y$

b) $f(x_0+\Delta x, y_0+\Delta y) \approx L(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + f_x\Big|_{(x_0,y_0)}\Delta x + f_y\Big|_{(x_0,y_0)}\Delta y$

$= 1 + \frac{2}{3}\Delta x + \frac{1}{3}\Delta y$. Use $\Delta x = -0.01$, $\Delta y = +0.01$.

$= 1 + \frac{-2/3}{100} + \frac{1/3}{100} = \underline{1 - \frac{1}{300}} \approx \underline{0.9967}$

cf exact $f(0.99, 2.0) = 0.996695\ldots$

$x^2+y^2+z^2 = 1$ equal ⇒ constraint, not $\le$ which defines a domain.

3) Constrained minima, maxima → use Lagrange multipliers, $g(x,y,z) = x^2+y^2+z^2$.

$$\boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x,y,z) &= 1 \end{aligned}}
\quad \begin{aligned} &\to \quad 4x = \lambda \cdot 2x \\ &\to \quad 2y-2 = \lambda \cdot 2y \\ &\to \quad -2z = \lambda \cdot 2z. \longrightarrow (\lambda+1)z = 0 \Rightarrow \lambda=-1 \text{ or } z=0. \end{aligned}$$

Keeping track of all solution possibilities is tricky.

$\lambda=-1$: $4x = -2x$ so $x=0$

$2y-2 = -2y$ so $y=\frac{1}{2}$.

find z using $g=1$: $0^2+\frac{1}{2}^2+z^2 = 1 \Rightarrow z = \sqrt{1-\frac{1}{4}} = \frac{\pm\sqrt{3}}{2}$

At these points $\left(0, \frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$, $f = 0+\frac{1}{4}-\frac{3}{4}-1 = -\frac{3}{2}$

$z=0$: multiply "x" & "y" eqns by y & x respectively:

$$\left.\begin{aligned} 4xy &= \lambda \cdot 2xy \\ 2xy-2x &= \lambda \cdot 2yx. \end{aligned}\right\} \text{RHS equal} \Rightarrow 4xy = 2xy - 2x.$$

$\Rightarrow 2x(y+1) = 0$

$\Rightarrow x=0$ or $y=1$.

$x=0$: $g=1$ gives $y=\pm 1$. $y=1$: $g=1$ gives $x=0$.

At point $(0,1,0)$ $f = -1$

$(0,-1,0)$ $f = +3$.

So abs. max is $+3$ at $(0,-1,0)$

abs. mins. are $-3/2$ at $\left(0, \frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$

4). Laplace's eqn means $u_{xx} = -u_{yy}$. so if one is ≥ 0, other is ≤ 0.

2nd derivs. test is $D = \underbrace{u_{xx}u_{yy}}_{-(u_{xx})^2} - (u_{xy})^2$. and we're told not all 2nd derivs are zero.

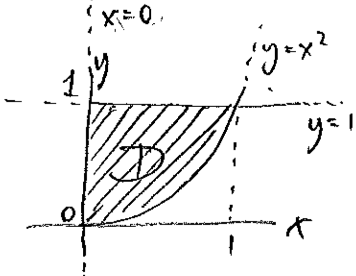
$\Rightarrow D < 0$.

If $D<0$ it's a saddle-point,

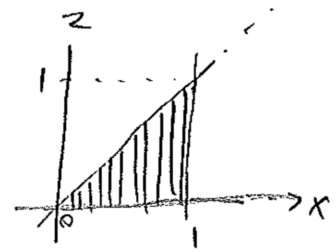
so it can't be a max or min $\Rightarrow$ $\boxed{\text{No}}$.

(this result is useful in electrostatics!)

5)

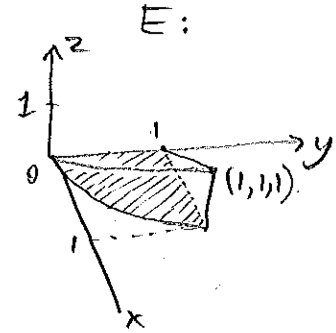


shadow in xy plane.



shadow in xz plane.

E:



do as Type 1, Type I.

↳ this gives limits on x as 0 to 1, which can be used to limit xz shadow.

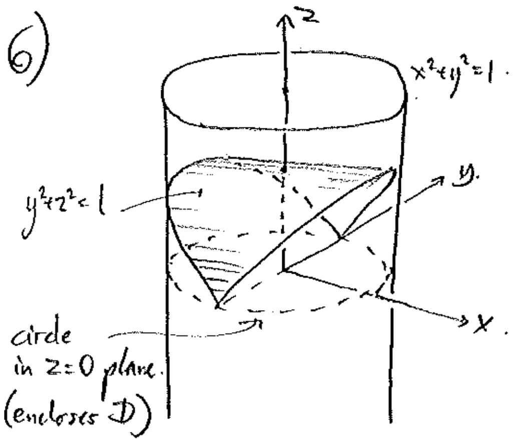
Inner. $A(x,y) = \int_{z=0}^{x} f(x,y,z)\,dz = \int_0^x z\,dz = \left[\frac{z^2}{2}\right]_0^x = \frac{x^2}{2}$

$I = \int_{x=0}^{1} \underbrace{\int_{y=x^2}^{1} A(x,y)\,dy}\,dx \rightarrow \left[\frac{yx^2}{2}\right]_{y=x^2}^{1} = -\frac{x^4}{2} + \frac{x^2}{2}$

$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2}\right)dx = \left[\frac{x^3}{6} - \frac{x^5}{10}\right]_0^1 = \frac{1}{6} - \frac{1}{10} = \frac{5-3}{30} = \underline{\frac{1}{15}}$

[or you can do as $I = \int_0^1 \int_0^{\sqrt{y}} \int_0^x z\,dz\,dx\,dy$, as Type II].

6)



$z = f(x,y) = +\sqrt{1-y^2}$ $\Rightarrow$ $f_x = 0$

$f_y = -2y \cdot \frac{1}{2}(1-y^2)^{-1/2}$

$= -\frac{y}{\sqrt{1-y^2}}$

↖ if you work over xy as domain, you need a height function f.

S.A. =

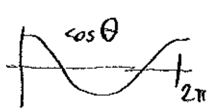
$\iint_D \sqrt{1 + f_x^2 + f_y^2}\,dA = \iint_D \sqrt{\frac{1-y^2}{1-y^2} + \frac{y^2}{1-y^2}}\,dA = \iint_D \frac{1}{\sqrt{1-y^2}}\,dA$

Domain in xy plane $= D =$ unit disc.

Use polars : $y = r\sin\theta$.

$S.A. = \int_0^{2\pi} \underbrace{\int_0^1 \frac{1}{\sqrt{1-r^2\sin^2\theta}} \overbrace{r\,dr\,d\theta}^{dA}}$

$\frac{d}{dr}\sqrt{1-r^2\sin^2\theta} = \frac{-2r\sin^2\theta\cdot\frac{1}{2}}{\sqrt{1-r^2\sin^2\theta}}$



$A(\theta) = \frac{-1}{\sin^2\theta}\left[\sqrt{1-r^2\sin^2\theta}\right]_{r=0}^{1}$

$= \frac{1}{\sin^2\theta}\left(1 - \sqrt{1-\sin^2\theta}\right) = \frac{1-|\cos\theta|}{\sin^2\theta}$

$S.A. = \int_0^{2\pi} \frac{1-|\cos\theta|}{\sin^2\theta}\,d\theta = 4\int_0^{\pi/2} \frac{1-\cos\theta}{\sin^2\theta}\,d\theta = 8\int_0^{\pi/2} \frac{\cancel{\sin^2(\theta/2)}}{2^2\cancel{\sin^2(\theta/2)}\cos^2(\theta/2)}\,d\theta$ ← double angle formulae.

$= 2\int_0^{\pi/2} \sec^2(\theta/2)\,d\theta = 2\left[2\tan\theta/2\right]_0^{\pi/2} = 4(\underbrace{\tan\tfrac{\pi}{4}}_{1} - \tan 0) = \boxed{4}$

Last bit was tricky.