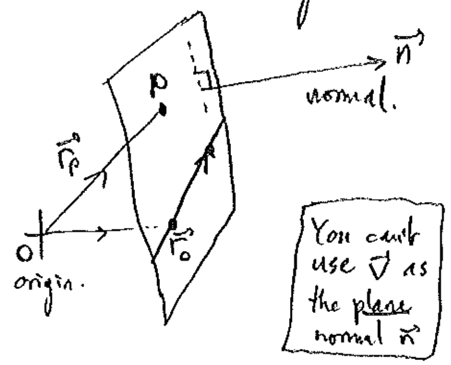
Midterm 1 Answers : CALCULUS III . NYU 2003 .

1. Draw a diagram!



To find $\vec{n}$, you need two vectors in the plane.

i) $\vec{v}$ of the line is one.:

line : $\vec{r}(t) = \vec{r_0} + \vec{v}t$, where $\vec{r_0} = (2,0,1)$ and $\vec{v} = (1,-5,-3)$

ii) Vector from $\vec{r_0}$ to P is another :

this is $\vec{r_p} - \vec{r_0} = (1,4,-2) - (2,0,1) = (-1,4,-3)$

Normal $\vec{n} = (1,-5,-3) \times (-1,4,-3) = (-27,-6,1)$

Need find any point in plane : you have 2 already ! ($\vec{r_0}$ and P).

Lets use $\vec{r_0}$:

Plane: $\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \Rightarrow -27x - 6y + z - \vec{n}\cdot\vec{r_0} = 0$

dot product is $-54 + 0 + 1 = -53$

$$\Rightarrow \underline{-27x - 6y + z + 53 = 0}$$

2. a) velocity $\vec{r}'(t) = (x'(t), y'(t))$

$\frac{d}{dt}(e^t \sin t) = e^t \cos t + e^t \sin t$ since $\frac{d}{dt}(e^t) = e^t$.

$\frac{d}{dt}(e^t \cos t) = e^t(-\sin t) + e^t \cos t$

$$= (e^t[\cos t - \sin t], e^t[\cos t + \sin t])$$

At $t=0$ this gives, using $e^0 = 1$, $\vec{r}'(0) = (1,1)$

speed $|\vec{r}'(t)| = \sqrt{x'(t)^2 + y'(t)^2} = \sqrt{e^{2t}(\cos^2 t - 2\sin t\cos t + \sin^2 t) + e^{2t}(\cos^2 t + 2\sin t\cos t + \sin^2 t)}$

$$= \sqrt{e^{2t} \cdot 2(\cos^2 t + \sin^2 t)} = \sqrt{2}\sqrt{e^{2t}} = \sqrt{2}e^t$$

At $t=0$ gives $|\vec{r}'(0)| = \sqrt{2}$

b) e^t is never zero. Can cross x-axis when $y(t) = 0$: $e^t \sin t = 0 \Rightarrow t = \frac{\pi}{2}$(odd integer)

" " y-axis " $x(t) = 0$: $e^t \cos t = 0 \Rightarrow t = \pi/2$ (even integer).

So any integer multiple of $\frac{\pi}{2}$ causes a crossing.

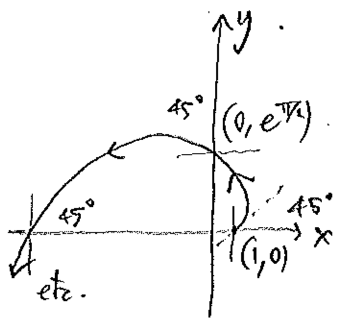
Angles? tangent slope $\frac{dy}{dx} = \frac{y'(t)}{x'(t)}$

$t = \frac{\pi}{2}$(odd integer) gives $x'(t) = e^t(1-0) = e^t$, $y'(t) = e^t(1+0) = e^t$ } $\frac{dy}{dx} = 1$ so $\theta = \tan^{-1}(1) = 45°$.

$t = \frac{\pi}{2}$(even integer) gives $x'(t) = \pm e^t$, $y'(t) = \pm e^t$ } $\frac{dy}{dx} = \pm 1$, $\theta = \tan^{-1}(\pm 1) = \pm 45°$.

Always crosses at 45°.

c) Sketch: Although looks like circle, cannot be since radius isn't const $\Rightarrow$ must be a spiral.

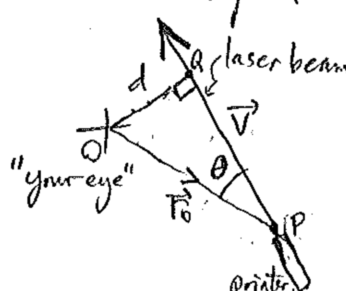


'Distance covered' ≡ length along the curve!

d) $L = \int_0^2 dt \cdot \text{speed} = \int_0^2 dt\, |\vec{r}'(t)|$

$= \sqrt{2}\int_0^2 dt\, e^t = \sqrt{2}\,[e^t]_0^2 = \underline{\sqrt{2}(e^2 - 1)}$

3. Draw a diagram.



Straight Line is always $\vec{r}(t) = \vec{r_0} + \vec{v}t$ [Don't confuse with plane eqn!]

We're given $\vec{r_0} = (1, 2, -1)$ and $\vec{v} = (1, -1, 2)$

Since $\vec{v}$ is not parallel to $\vec{r_0}$, you can tell the beam won't hit you in the eye.

Explicitly, let's solve for t when $x = 0$: $0 = x(t) = 1 + 1\cdot t \Rightarrow t = -1$

$y = 0$: $0 = y(t) = 2 - 1\cdot t \Rightarrow t = 2$

These differ so at no point can the beam reach $x = 0, y = 0$, let alone reach the origin $(0,0,0)$.

We want perpendicular distance d [$\neq$ distance from a plane!].

Four ways you could solve this:

[A] trig: $d = |\vec{r_0}|\sin\theta$

use $|\vec{r_0} \times \vec{v}| = |\vec{v}||\vec{r_0}|\sin\theta = |\vec{v}|d$.

$\Rightarrow d = \frac{|\vec{r_0}\times\vec{v}|}{|\vec{v}|} = \frac{|(3, -3, -3)|}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{3\sqrt{3}}{\sqrt{6}} = \frac{3}{\sqrt{2}}$

$\lvert PQ\rvert = -(\text{component of } \vec{r_0} \text{ along } \vec{v}) = -\frac{\vec{r_0}\cdot\vec{v}}{\lvert\vec{v}\rvert}$
only used in method [B].

$= \frac{-(1-2-2)}{\sqrt{1^2+1^2+2^2}} = \frac{3}{\sqrt{6}}$

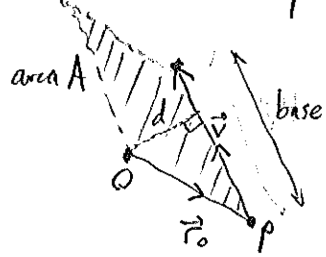
B: You can build vector $\vec{PQ} = |PQ| \cdot \frac{\vec{v}}{|\vec{v}|}$ ← unit vector along line.

$$= \frac{3}{\sqrt{6}} \cdot \frac{(1,-1,2)}{\sqrt{6}} = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 1\right)$$

Now $d = |OQ| = |\vec{r_0} + \vec{PQ}| = \left|(1,2,-1) + \left(\tfrac{1}{2},-\tfrac{1}{2},1\right)\right|$ [not so nice a way.]

$$= \left|\left(\tfrac{3}{2}, \tfrac{3}{2}, 0\right)\right| = \tfrac{3}{\sqrt{2}}.$$

C: Remember area of parallelogram $A = |\vec{r_0} \times \vec{v}| = \underbrace{\text{base}}_{|\vec{v}|} \times \underbrace{\text{height}}_{d\text{, we want}}$



$$\Rightarrow d = \frac{|\vec{r_0} \times \vec{v}|}{|\vec{v}|} = \frac{|(3,-3,-3)|}{\sqrt{6}} = \frac{3}{\sqrt{2}}.$$

D: ← If all else fails!

Write distance to origin as func. of t, and minimize it (not intended method!)

$$d(t) = |\vec{r_0} + \vec{v}t| = \sqrt{(1+t)^2 + (2-t)^2 + (-1+2t)^2}$$
$$= \sqrt{4 - 6t + 6t^2}$$

$$\frac{d}{dt} d(t) = \tfrac{1}{2}(-6+12t)(4-6t+6t^2)^{-1/2} = 0 \quad \text{when} \quad (-6+12t) = 0, \; t = \tfrac{1}{2}.$$

Location at $t = 1/2$ is $(3/2, 3/2, 0)$

$$\Rightarrow \text{distance } d(t=\tfrac{1}{2}) \text{ is } \sqrt{2} \cdot \tfrac{3}{2} = \tfrac{3}{\sqrt{2}}$$

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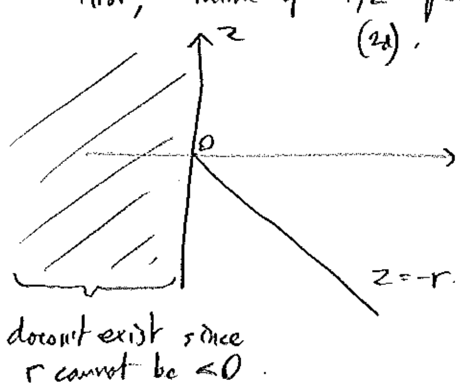
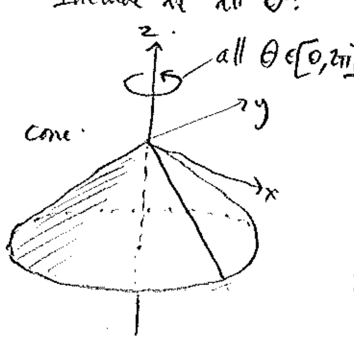
4. Draw x, y, z axes since there is no fixed "r-axis" or "θ-axis" in cylindrical coords.

First, think of r, z plane: (2d).

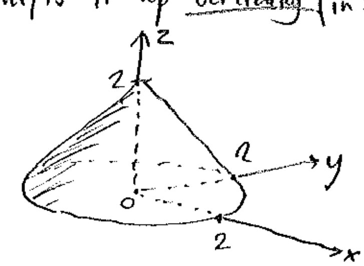


3d: Include at all θ:



we don't want $z<0$ part.

Add 2 to all z values: shifts it up vertically (in z):



45° cone of height 2.

5.

Use vector version of Fundamental Theorem of Calculus

$$\vec{r}(t_B) - \vec{r}(t_A) = \int_{t_A}^{t_B} \vec{r}\,'(t)\,dt$$

we want $t_A = 0$ (starting time) and $t_B =$ "t", our general time.

Rearrange $\vec{r}(t) = \vec{r}(0) + \int_0^t \vec{r}\,'(t')\,dt'$

$\vec{r}(0)$ ← start location

← notice we use integration variable t' now since we've already 'used up' the symbol t.

Components:

$$x(t) = x_0 + \int_0^t \frac{dt'}{1+t'} = 3 + \left[\ln(1+t')\right]_0^t = 3 + \ln(1+t) - \ln 1$$

($x_0 = 3$, $\ln 1 = 0$.)

$$y(t) = y_0 + \int_0^t (1+t')\,dt' = 3 + \left[t' + \tfrac{1}{2}t'^2\right]_0^t = 3 + t + \tfrac{1}{2}t^2$$

($y_0 = 3$)

$$z(t) = z_0 - \int_0^t \sin 2t'\,dt' = 3 - \left[-\tfrac{1}{2}\cos 2t'\right]_0^t$$

$$= 3 + \tfrac{1}{2}(\cos 2t - \cos 0)$$

$$= 5/2 + \tfrac{1}{2}\cos 2t.$$

($z_0 = 3$)

Write in terms of $\hat{i}\,\hat{j}\,\hat{k}$ is $\vec{r}(t) = \hat{i}x(t) + \hat{j}y(t) + \hat{k}z(t)$

$$= \hat{i}\left(3 + \ln(1+t)\right) + \hat{j}\left(3 + t + \tfrac{1}{2}t^2\right) + \hat{k}\left(5/2 + \tfrac{1}{2}\cos 2t\right)$$

or could write $\left(5/2 + \tfrac{1}{2}\cos 2t\right)$ as $3 - \sin^2 t$