1. (10) A rectangular box is to hold 16 cubic meters. The material used for the top and bottom of the box is twice as expensive per square meter than the material used for the remaining four sides. Find the dimensions of the box that minimizes the cost of the material used in the box.

ANS: We want to minimize $C(x, y, z) = 4xy+2xz+2yz$ subject to the constraint $g(x, y, z) = xyz = 16$. Since the cost clearly becomes infinite if any of the dimensions tends to zero or grows without bound, there is clearly a minimum value. Therefore one acceptable method is to use Lagrange multipliers. Thus we want to solve for x, y, z, λ in the equations $\nabla C = \lambda \nabla g$ and $g = 16$. This gives us the simultaneous equations

$$\begin{aligned} 4y + 2z &= \lambda yz \\ 4x + 2z &= \lambda xz \\ 2x + 2y &= \lambda xy \\ xyz &= 16. \end{aligned}$$

Since x, y and z must all be nonzero, we can multiply the first equation by x and the second by y to obtain

$$4xy + 2xz = 4xy + 2yz.$$

Since $z \neq 0$, we must have $x = y$.

Similarly, multiplying the second equation by y and the third by z gives

$$4xy + 2yz = 2xz + 2yz.$$

Since $x \neq 0$, we must have $z = 2y$.

Plugging this information into the fourth equation gives $x(x)(2x) = 16$, or $x = 2$. Therefore the dimensions that give the minimum cost are $\boxed{2\ m \times 2\ m \times 4\ m}$.

Dimensions = ______________________

2. (20) Let E be the solid lying in the first octant, inside the cone $z = \sqrt{x^2 + y^2}$ and below the plane $z = 2$. (That is, E is the set of (x, y, z) such that $x^2 + y^2 \geq z^2$, $x \geq 0$, $y \geq 0$ and $z \leq 2$.) We want to consider the triple integral

$$\iiint_E xyz\, dV.$$

(a) Express $\iiint_E xyz\, dV$ as an iterated integral in rectangular coordinates.

ANS: We can view E as the solid region between the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 2$ which is above the planar region D in the xy-plane which is the quarter of the circle of radius 2 (centered at the origin) lying in the first quadrant. Hence the integral equals $\boxed{\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 xyz\, dz\, dy\, dx}$

(Your answer should be an iterated integral!)

(b) Express $\iiint_E xyz\,dV$ as an iterated integral in cylindrical coordinates.

ANS: $\boxed{\int_0^{\pi/2}\int_0^2\int_r^2 r^3\cos(\theta)\sin(\theta)z\,dz\,dr\,d\theta}$

(Your answer should be an iterated integral!)

(c) Express $\iiint_E xyz\,dV$ as an iterated integral in spherical coordinates.

ANS: $\boxed{\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{2\sec(\varphi)} \rho^5\sin^3(\varphi)\cos(\varphi)\cos(\theta)\sin(\theta)\,d\rho\,d\theta\,d\varphi.}$

(Your answer should be an iterated integral!)

(d) Evaluate $\iiint_E xyz\,dV$

ANS: The most reasonable approach seems to be cylindrical coordinates. Then

$$\int_0^{\pi/2}\int_0^2\int_r^2 r^3\cos(\theta)\sin(\theta)z\,dz\,dr\,d\theta = \Big(\int_0^{\pi/2}\sin(\theta)\cos(\theta)\,d\theta\Big)\Big(\frac{1}{2}\int_0^2(4-r^2)r^3\,dr\Big)$$

which, after letting $u=\sin(\theta)$ in the first integral, is

$$\begin{aligned} &= \frac{1}{2}\Big(\int_0^1 u\,du\Big)\Big(r^4-\frac{r^6}{6}\Big)_0^2 \\ &= \frac{1}{2^2}\Big(2^4-\frac{2^6}{6}\Big) \\ &= 4-\frac{8}{3} = \boxed{\frac{4}{3}}. \end{aligned}$$

3. (10) Find the maximum and minimum values of $f(x, y) = x^2 + x + y^2$ on and inside the ellipse $2x^2 + y^2 = 2$.

ANS: Since the ellipse and its interior form a *closed and bounded* subset $D = \{\,(x, y) : 2x^2+y^2 \le 2\,\}$ of the plane, f must attain its max and min on D.

We first check for critical points in the interior of the ellipse. Since $f_x(x, y) = 2x + 1$ and $f_y(x, y) = 2y$, the only critical point occurs at $(-\frac{1}{2}, 0)$. Note that $f(-\frac{1}{2}, 0) = -\frac{1}{4}$.

To check for critical points on the boundary, we'll use Lagrange multipliers with $f(x, y)$ as above and with constraint $g(x, y) = 2x^2 + y^2 = 2$. The equations $\nabla f = \lambda \nabla g$ and $g = 1$ give us the simultaneous equations

$$\begin{aligned} 2x + 1 &= 4\lambda x \\ 2y &= 2\lambda y \\ 2x^2 + y^2 &= 2. \end{aligned}$$

From the second equation, we see that either $y = 0$ or $\lambda = 1$. If $y = 0$, then we plug into the third equation to get two critical points $(\pm 1, 0)$. If $\lambda = 1$, then we plug into the first equation to get $x = \frac{1}{2}$. Then plugging into the third equation gives us two more critical points $(\frac{1}{2}, \pm\sqrt{\frac{3}{2}})$.

Now we compare values. First, $f(\frac{1}{2}, \pm\sqrt{\frac{3}{2}}) = \frac{1}{4} + \frac{1}{2} + \frac{3}{2} = \frac{9}{4}$. Secondly, $f(1, 0) = 2$. Thirdly, $f(-1, 0) = 0$.

Comparing these values with the values of f at its interior critical points, the maximum of f on D is given by $\boxed{\frac{9}{4} \text{ which occurs at } (\frac{1}{2}, \pm\sqrt{\frac{3}{2}})}$, and the minimum is $\boxed{-\frac{1}{4} \text{ which occurs at } (-\frac{1}{2}, 0)}$.

The maximum is ____________ and occurs at the point(s) ______________________

The minimum is ____________ and occurs at the point(s) ______________________

4. (10) Find the maximum value of

$$\iiint_E (1 - x^2 - y^2 - z^2)\,dV$$

over all solid regions E in $\mathbf{R}^3$. Justify your assertions.

ANS: Note that the integrand is negative unless (x, y, z) is a point on or inside the unit sphere $x^2 + y^2 + z^2 = 1$. Thus, if E is any region in $\mathbf{R}^3$ and if E_1 is the part of E on and inside the sphere and E_2 the part outside, then

$$\begin{aligned} \iiint_E (1 - x^2 - y^2 - z^2)\,dV &= \iiint_{E_1} (1 - x^2 - y^2 - z^2)\,dV + \iiint_{E_2} (1 - x^2 - y^2 - z^2)\,dV \\ &\le \iiint_{E_1} (1 - x^2 - y^2 - z^2)\,dV. \end{aligned}$$

Thus it is clear that to get the maximum possible value, we need E contained inside the unit sphere.

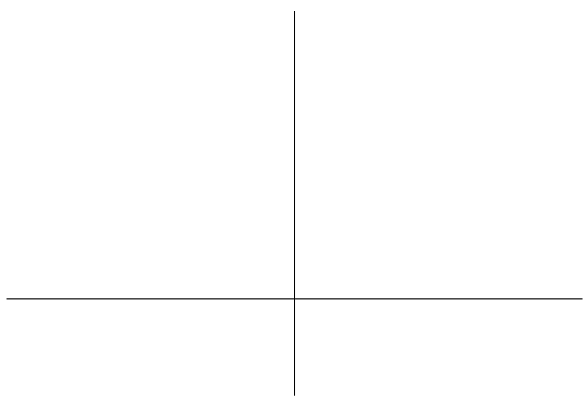
Now if we think of the value of the integral as the mass of a solid with positive density, then it is clear that the mass of any region inside of the unit sphere is larger than the mass of any region it contains. Hence the maximum value is attained when E is all of $\{\,(x,y,z) : x^2+y^2+z^2 \le 1\,\}$. Then, using spherical coordinates, the maximum is

$$\begin{aligned}
\iiint_E (1-x^2-y^2-z^2)\,dV &= \int_0^\pi \int_0^{2\pi} \int_0^1 (1-\rho^2)\rho^2 \sin(\varphi)\,d\rho\,d\theta\,d\varphi \\
&= 2\pi\Big(\int_0^\pi \sin(\varphi)\,d\varphi\Big)\Big(\int_0^1 (\rho^2-\rho^4)\,d\rho\Big) \\
&= 2\pi(2)\big(\frac{1}{3}-\frac{1}{5}\big) \\
&= \boxed{\frac{8\pi}{15}}.
\end{aligned}$$

Max = ______________________

5. (15) Consider the iterated integral $\displaystyle\int_1^2 \int_0^y \frac{1}{(\sqrt{x^2+y^2})^3}\,dx\,dy$.

(a) Sketch the region D such that $\displaystyle\iint_D \frac{1}{(\sqrt{x^2+y^2})^3}\,dA = \int_1^2 \int_0^y \frac{1}{(\sqrt{x^2+y^2})^3}\,dx\,dy$.



(b) Reverse the order of integration in the above iterated integral. Your answer may be a sum of iterated integrals. (DO NOT evaluate any integrals in this part.)

ANS: $\boxed{\displaystyle\int_0^1 \int_1^2 \frac{1}{(\sqrt{x^2+y^2})^3}\,dy\,dx + \int_1^2 \int_x^2 \frac{1}{(\sqrt{x^2+y^2})^3}\,dy\,dx}$

(Your answer should be an iterated integral!)

(c) Express $\displaystyle\iint_D \frac{1}{(\sqrt{x^2+y^2})^3}\,dA$ as an iterated integral in polar coordinates.

ANS: The line $y = b$ in polar coordinates is given by $r = b\csc(\theta)$. The integrand converts to $1/r^3$. Then, remembering to add in the integrating factor "r", we have $\boxed{\displaystyle\int_{\pi/4}^{\pi/2}\int_{\csc(\theta)}^{2\csc(\theta)} \frac{1}{r^2}\,dr\,d\theta}$.

(Your answer should be an iterated integral!)

(d) Evaluate $\displaystyle\iint_D \frac{1}{(\sqrt{x^2+y^2})^3}\,dA$.

ANS: We'll use polar coordinates:

$$\begin{aligned}\int_{\pi/4}^{\pi/2}\int_{\csc(\theta)}^{2\csc(\theta)} \frac{1}{r^2}\,dr\,d\theta &= \int_{\pi/4}^{\pi/2}\left(-\frac{1}{r}\right)\Big|_{\csc(\theta)}^{2\csc(\theta)}\,d\theta \\ &= \frac{1}{2}\int_{\pi/4}^{\pi/2}\sin(\theta)\,d\theta \\ &= \frac{1}{2}\Big(-\cos(\theta)\Big)\Big|_{\pi/2}^{\pi/4} \\ &= \boxed{\frac{1}{2\sqrt{2}}}.\end{aligned}$$

6. (20) SHORT ANSWER: Place your answer carefully in the space provided. On this question, you do not have to show your work and no partial credit will be assigned.

(a) Evaluate $\displaystyle\int_{-1}^{0}\int_{0}^{\sqrt{1-y^2}}(x^2+y^2)\,dx\,dy$.

ANS: Use polar coordinates: $\int_{-\pi/2}^{0}\int_0^1 r^3\,dr\,d\theta = \left(\frac{\pi}{2}\right)\left(\frac{1}{4}\right) = \boxed{\frac{\pi}{8}}$.

(b) Let E be the solid region in the ***first octant*** inside of both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 2$. Find the value of $\iiint_E z\,dV$.

ANS: Use cylindrical coordinates: $\int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{2-r^2}} zr\,dz\,dr\,d\theta = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right) \int_0^1 r(2-r^2)\,dr = \boxed{\frac{3\pi}{16}}$.

(c) Find the value of $\int_{-1}^{1} \int_0^1 xe^{y^2}\,dy\,dx$.

ANS: The integral equals $\left(\int_0^1 e^{y^2}\,dy\right) \cdot \left(\int_{-1}^1 x\,dx\right) = \left(\int_0^1 e^{y^2}\,dy\right) \cdot 0 = \boxed{0}$.

(d) Find the volume of the solid region bounded by the parabolic cylinder $x = z^2$ and the planes $x = z + 2$, $y = 1$ and $y = 2$.

ANS: $V = \int_{-1}^2 \int_{z^2}^{z+2} \int_1^2 1\,dy\,dx\,dz = \int_{-1}^2 (z + 2 - z^2)\,dz = 2 + 4 - \frac{8}{3} - (\frac{1}{2} - 2 + \frac{1}{3}) = \boxed{\frac{9}{2}}$.

(e) Compute $\iiint_E (x^2 + y^2 + z^2)\,dV$ if E is the region bounded by the cylinder $x^2 + y^2 = a^2$, the xy-plane and the plane $z = 2$.

ANS: In cylindrical coordinates: $\int_0^{2\pi} \int_0^a \int_0^2 (r^2 + z^2) r\,dz\,dr\,d\theta = 2\pi \int_0^a (2r^3 + \frac{8}{3}r)\,dr = \boxed{2\pi(a^4/2 + \frac{4}{3}a^2)}$.

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7. (15) MULTIPLE CHOICE. Circle the best response. No partial credit will be given on this problem and you do not need to justify your answers.

(a) The function $f(x, y) = x^2 + axy + y^2$ has a critical point at $(0, 0)$, and f has a local minimum at $(0, 0)$ if ...

A. $a < 2$ **B**. $a < 0$ **C**. $a > 2$ or $a < -2$ **D**. $-2 < a < 2$

E. $a > 0$

ANS: Easy computations show that $D(0,0) = 4 - a^2$. For $D > 0$, we need $|a| < 2$. (Note that if $|a| = 2$, then the second derivative test gives no information, but that $(0,0)$ is a local minimum (global, in fact) in that case.) Since we have $f_{xx}(0,0) = 2 > 0$, the correct answer is **D**.

(b) The function $f(x, y) = x^3 + 2xy + y^2$ has two critical points. The function f has ...

A. A saddle point and a local minimum **B**. A local max and a local min

C. A saddle point and a local maximum **D**. Two local minima

E. None of these

ANS: We compute that $f_x(x, y) = 3x^2 + 2y$ and $f_y(x, y) = 2x + 2y$. Setting f_y equal to zero forces $x = -y$. Plugging into the first equation given either $x = 0$ or $x = \frac{2}{3}$. Thus, the two critical points are $(0,0)$ and $(\frac{2}{3}, -\frac{2}{3})$. Since $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = 2$ and $f_{yy}(x, y) = 2$, $D(0,0) < 0$ and $(0,0)$ is a saddle point. On the other hand, $D(\frac{2}{3}, -\frac{2}{3}) > 0$ and $f_{xx}(\frac{2}{3}, -\frac{2}{3}) > 0$, so $(\frac{2}{3}, -\frac{2}{3})$ is a local minimum. Thus the correct answer is **A**.

(c) Using spherical coordinates, the cone $z = \sqrt{3x^2 + 3y^2}$ is given by ...

A. $\varphi = \pi/3$ **B**. $\theta = \pi/6$ **C**. $\varphi = 3\rho$ **D**. $\varphi = \theta$

E. None of these

ANS: Since $x^2 + y^2 = \rho^2 \sin(\varphi)^2$, and since $\rho \geq 0$ and $\sin(\varphi) \geq 0$, we must have $\tan(\varphi) = \frac{1}{\sqrt{3}}$. Thus we have $\varphi = \frac{\pi}{6}$, and the correct answer is **E**.

(d) The surface described in spherical coordinates by $\rho \cos \varphi = a$, where a is a positive constant, can also be described in a different coordinate system by setting one variable equal to a constant, as follows ...

A. Setting z equal to a constant in rectangular coordinates.
B. Setting θ equal to a constant in cylindrical coordinates.
C. Setting r equal to a constant in cylindrical coordinates.
D. Setting x equal to a constant in rectangular coordinates.
E. None of the above.

ANS: In spherical coordinates, $z = \rho\cos(\varphi)$. Therefore the correct answer is **A**.

(e) The iterated integral $\displaystyle\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sec\varphi} \rho^3\sin^2\varphi\, d\rho\, d\theta\, d\varphi$ is equal to

A. $\displaystyle\int_0^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 \sqrt{x^2+y^2+z^2}\, dz\, dy\, dx$

B. $\displaystyle\int_0^{2\pi}\int_0^1\int_r^1 r^2\, dz\, dr\, d\theta$

C. $\displaystyle\int_0^{2\pi}\int_0^1\int_0^{\sqrt{3}z} r^2\, dr\, dz\, d\theta$

D. $\displaystyle\int_0^1\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2-x^2}}^{\sqrt{1-z^2-x^2}} y\, dy\, dx\, dz$

E. None of These

ANS: In rectangular coordinates, the integrand is $\rho\sin(\varphi) = \sqrt{x^2+y^2}$. The region is that inside the cone $z = \sqrt{x^2+y^2}$ and below the plane $z = 1$. Consequently, the correct answer is **B**.

NAME : ______________________________

SECTION : (circle one) Groszek Mileti Williams

Math 11

10 November 2008

Exam II

Instructions: This is a closed book, closed notes exam. You are not to provide or receive help from any outside source during the exam except that you may ask any of the instructors for clarification. You have two hours and you should attempt all 7 problems.

- Wait for signal to begin.
- ***Print*** your name in the space provided and circle your instructor's name.
- Sign the FERPA release below *only if* you wish your exam returned via the homework boxes.
- Calculators or other computing devices are not allowed.
- Except for problems #6 and #7, you must show your work and justify your assertions to receive full credit.
- *Place your final answer in the space provided*!

FERPA RELEASE: Because of privacy concerns, we are not allowed to return your graded exams to the homework boxes without your permission. If you wish us to return your exam to your homework box, please sign on the line indicated below. Otherwise, you will have to pick your exam up in your instructor's office after lecture in which the exams are returned.

SIGN HERE: ____.

Problem	Points	Score
1	10	
2	20	
3	10	
4	10	
5	15	
6	20	
7	15	
Total	100	