V63.0123-1 : Calculus III. Final. Spring 2003

You have 110 minutes. Non-graphing calculators and a double-sided equation sheet are allowed. You only need answer 7 of the 8 questions. Please indicate at the top which one you did NOT attempt.

1. [10 points]

   Find the critical points of $f(x, y) = x^2 + y^2 + x^2y + 4$. For each point found specify if it is a local minimum, local maximum, or saddle point.

2. [10 points]

   For a general 3d vector function $\mathbf{r}(t)$, prove the following two (unrelated) facts. [As usual, the prime symbol means t-derivative].

   (a) $$\frac{d}{dt}(\mathbf{r}' \times \mathbf{r}) \;=\; \mathbf{r}'' \times \mathbf{r}.$$

   (b) $$\frac{d}{dt}|\mathbf{r}| \;=\; \frac{1}{|\mathbf{r}|}\mathbf{r} \cdot \mathbf{r}'.$$ [Hint: components, tree of dependence]

3. [10 points]

   Find the volume that lies inside both the sphere $x^2 + y^2 + z^2 = 4$ and the cylinder $x^2 + y^2 = 1$.

4. [10 points]

   S is the surface of the hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$, oriented upwards, combined with the disc $x^2 + y^2 \leq 9$ lying in the plane $z = 0$, oriented downwards. Find the flux through S of the vector field $\mathbf{F} = (z,\, yx + z,\, z^2/6 + y)$. [Hint: Divergence Theorem].

5. [10 points]

   Find $\int_C \frac{1}{2}y^2 \sin x\, dx \;+\; (1 - y\cos x)dy$ where C is the planar curve defined parametrically by $x(t) = \pi t/2$ and $y(t) = \pi t^3/2$, with t starting at 0 and ending at 1. [Hint: Fundamental Theorem of Line Integrals].

6. [10 points]

   C is the boundary curve of the piece of the plane $x + y + z = 1$ which lies in the first octant (*i.e.* the region $x \geq 0$, $y \geq 0$, $z \geq 0$), traversed in a counter-clockwise sense when viewed from above. Compute $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the field $F = xz\mathbf{i} + (y^2 - \frac{1}{2}z^2)\mathbf{j} + z^2\mathbf{k}$. [Hint: Stokes].

7. [10 points]

   The curve C is defined by $x = t^3/3$, $y = t^2/2$, with domain $t \in [0, 1]$.

   (a) Find the arc length of C.

   (b) Evaluate the line integral $\int_C f\, ds$ of the function $f(x, y) = 1/\sqrt{1 + 2y}$.

8. [10 points]

   C is composed of three segments traversed in a clockwise sense: the straight line from $(0, 0)$ to $(0, 2)$, the quarter circle from there to $(2, 0)$, and the straight line returning to $(0, 0)$. Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the field $\mathbf{F} = (1 - x^2y)\mathbf{i} + y\mathbf{j}$. [Hint: Green's Theorem].