1. A note on line integrals

1.1. The fundamental theorem. If $f(x,y)$ is a two variable function, then ∇f is a vector field on $\mathbb{R}^2$. Suppose C is a curve in $\mathbb{R}^2$ with initial point (x_0, y_0) and terminal point (x_1, y_1). Then the *fundamental theorem for line integrals* says that

$$\int_C \nabla f(x,y) \cdot d\mathbf{r} = f(x_1, y_1) - f(x_0, y_0).$$

Notice that this is true regardless of singularities in the vector field ∇f.

The fundamental theorem is also true for functions $f(x,y,z)$ of three variables.

1.2. Conservative vector fields. If $\mathbf{F} = \langle P, Q \rangle$ is a vector field on $\mathbb{R}^2$, then we call $\mathbf{F}$ a *conservative vector field* if there exists a potential function $f(x,y)$ with $\mathbf{F} = \nabla f$.

If $\mathbf{F}$ is conservative, then we can apply the fundamental theorem to compute line integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(x_1, y_1) - f(x_0, y_0).$$

because in this case

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f(x,y) \cdot d\mathbf{r}.$$

It follows from this that line integrals for a conservative field $\mathbf{F}$ are *path independent.* this means that if C_1 and C_2 are two curves with the same initial and terminal points, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$

because both integrals are equal to $f(x_1, y_1) - f(x_0, y_0)$. A different way to say this is that if C is *any* closed curve in the domain of a conservative vector field $\mathbf{F}$, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Notice that these facts remain true regardless of the existence of critical points for $\mathbf{F}$, because the fundamental theorem has nothing to do with critical points!

1.3. Singularities. Now suppose we don't know whether a potential function $f(x,y)$ exists, but we *do* know (by direct verification) that

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

This is almost as good as being conservative, but since we do *not* know that $\mathbf{F} = \nabla f$ we can *not* apply the fundamental theorem. Nevertheless, if $\partial P/\partial y = \partial Q/\partial x$ then we still have path independence under certain extra conditions.

First, if C_1 and C_2 are two curves with the same end points, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r},$$

as long as the region bounded by the two curves does not contain any singularities of $\mathbf{F}$, i.e., $\mathbf{F}$ exists everywhere inside of the two curves. Likewise, if C is a closed curve, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0,$$

as long as the region bounded C does not contain any singularities.

Why is this true? Not because of the fundamental theorem, because it does not apply in this case. But it is true because of Green's theorem, which says that

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = 0,$$

because $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$.

Green's theorem only applies if $\mathbf{F}$ exists and is continuous on the domain D, so that the double integral $\iint_D$ makes sense. That is why no singularities are allowed in this case. In the case where $\mathbf{F}$ is conservative we apply the fundamental theorem instead, where singularities are irrelevant.

1.4. **No singularities.** Finally, suppose $\mathbf{F} = \langle P, Q \rangle$ is a vector field in $\mathbb{R}^2$ for which

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

is true. We have seen that such fields have almost the same properties as conservative fields, but that we have to take singularities into account.

Now, if $\mathbf{F}$ is a vector field that is defined on *all* of $\mathbb{R}^2$, and if there are *no* singularities whatsoever, then the condition

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

actually *implies* that $\mathbf{F}$ must be conservative. So in the absence of singularities, the condition $\partial P/\partial y = \partial Q/\partial x$ is equivalent to saying that $\mathbf{F} = \nabla f$, and you know for certain that such $f(x,y)$ will exist.

Notice that *even* if there are singularities, it *may* still be true that $\mathbf{F}$ is conservative. It is just that in this case it is not sufficient to check that $\partial P/\partial y = \partial Q/\partial x$.