Homework for Chapter 5: Due January 27

1. Given an n by n nonsingular matrix A, how do you efficiently solve the following problems using Gaussian elimination with partial pivoting?

   (a) Solve the linear system $A^k x = b$, k is a positive integer.

   (b) Compute $\alpha = c^T A^{-1} b$.

   (c) Solve the matrix equation $AX = B$, where B is n by m.

   You should (1) describe your algorithms, (2) present them in pseudo-code (using a MATLAB-like language) and (3) estimate the cost by giving the rquired flops.

2. Matrix A is called *strictly column diagonally dominant*, or diagonally dominant for short, if

$$|a_{ii}| > \sum_{j=1, j\neq i} |a_{ji}|.$$

   (a) Show that A is nonsingular. Hint: Use Gershgorin's theorem.

   (b) Show that Gaussian elimination with partial pivoting does not actually permute any rows, that is, it is identical to Gaussian elimination without pivoting. Hint: Show that after one step of Gaussian elimination, the trailing $(n-1)\times(n-1)$ submatrix, the Schur complement of a_{11} in A, is still diagonally dominant.

3. Consider the 2×2 linear system of equations

$$Ax = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b.$$

   We will use two methods to solve it.

   - **Algorithm 1:** Gaussian elimination with partial pivoting.
   - **Algorithm 2:** Cramer's rule

$$\begin{aligned} det &= a_{11} * a_{22} - a_{12} * a_{11}; \\ x_1 &= (a_{22} * b_1 - a_{12} * b_2)/det; \\ x_2 &= (-a_{21} * b_1 + a_{11} * b_2)/det; \end{aligned}$$

Show by means of a numerical example that Cramer's rule is not backward stable. Hint: Choose the matrix A to be neary singular and $b \approx \mathbf{a}_2$ (the second column of A). What does backward stability imply about the size of the residual?