M116 Lec 5.

- pick scribe date.
- OH. W 2, Th. 5-6

① Th. 4/2/14.

Today: feedback slips for stud. lecs. (& me?)

QR factorization. Goal: solve $Ax=b$ eg for $A \in \mathbb{C}^{m\times n}$ reach.

$m>n$: overdetermined ⇒ want least-squares: find Ax in range of A closest to b.

e.g.: Larry's research, data fitting for models, ...

idea: factorize $A = QR$ (R upper-tri)

Q o.n. cols $q_1 \cdots q_n$: o.n.b. for range of A.

(or first r cols are if R has $r<n$ diagonal entries)

App's of QR: Useful? [NLA §11]

$Q^*Ax = Q^*b$

$Q^*Q\,R$ ($=I$)

so $Rx = Q^*b$ ↷ Rhs computable in flops fast? $O(mn)$

Idea of matrix factorization is one 'top level' solve by back-substitution, flops fast? $O(n^2)$

"This also works for square systems" transformed $2n^3$ comp.

3 steps: i) compute factors $A=QR$ ⟶ we will show B.S. alg: ie gives $\tilde{Q}, \tilde{R}$ s.t. $\tilde{Q}\tilde{R} = A + \delta A$.

ii) Rhs $y = Q^*b$

iii) solve $Rx = y$ ↷ matrix: $\kappa(Q)=1$. Q unitary

for some $\frac{\|\delta A\|}{\|A\|} \le \mathcal{O}(\varepsilon_{mach})$

can prove B.S.

Each step B.S. ⇒ whole process is ↷ returns $\tilde{x}$ s.t. $(A+\delta A)\tilde{x} = b$ (ask what means?)

for some $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_{mach})$.

(NLA §16)

Recall: Given above B.S. alg for solving $Ax=b$, how accurate can expect soln. $\tilde{x}$ to be?

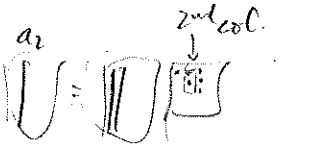
Thm (15.1) rel. err. $\xi := \frac{\|\tilde{x}-x\|}{\|x\|} = O(\kappa \varepsilon_{mach})$

for problem $\kappa(A)$

this is the best you can reasonably expect:

eg if $\kappa = 10^{10}$, only expect $\tilde{x}$ correct to 6 digits, but get $r = A\tilde{x} - b$ has $\|r\| \approx 10^{-16}$.

Doing QR: $A = [a_1 \cdots a_n]$ — 2nd col.



$A = QR$ says
$$a_1 = r_{11} q_1$$
$$a_2 = r_{12} q_1 + r_{22} q_2 \quad (*)$$
$$\vdots$$

$\{q_i\}_{i=1}^{j}$ is o.n.b. for 1st j cols of A — remind of?

Alg. 1: Gram-Schmidt.

for $j = 1 \cdots n$ ← is a working vector
  $v_j = a_j$
  for $i = 1 \cdots j-1$
    $v_j = v_j - (q_i^* a_j) q_i$
  end
  $q_j = v_j / \|v_j\|$ — call r_{jj} why? — call r_{ij} why? — see now...
end.

Your Check (run code on paper, using r_{ij}, see if agrees (*)):
$q_1 = a_1 / r_{11}$ (i loop zero times) $\longrightarrow$ $a_1 = r_{11} q_1$ ✓ agrees w/ QR
$q_2 = \dfrac{a_2 - r_{12} q_1}{r_{22}} \longrightarrow a_2 = r_{12} q_1 + r_{22} q_2$ ✓
$q_3 = \dfrac{a_3 - r_{13} q_1 - r_{23} q_2}{r_{33}}$ ✓
etc. So QR ↔ G-S keeping track of inner prod weights

demo play-decompsn → pick A w/ decaying singulars. → find $Q^*Q \neq I$! roundoff unstable. r_{jj} don't decay below 10^{-8}. Is B.S.? yes.

Alg. 2
init $v_i = a_i$ $\forall i = 1 \cdots n$
for $i = 1 \cdots n$, $q_i = v_i / \|v_i\|$ → call r_{ii}
  for $j = i+1, \cdots n$
    $v_j = v_j - q_i^* a_j \, q_i$ → r_{ij}
  end
end.

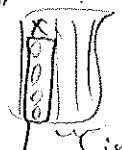
How relates — is width same or diff? "Run" it:
check: $i=1$: $q_1 = a_1 / r_{11}$ ✓ $\quad j=2$: $v_2 = a_2 - r_{12} q_1$ $\quad j=3$: $v_3 = a_3 - r_{13} q_1$, ... $v_n = a_n - r_{1n} q_1$
$i=2$: $q_2 = v_2 / r_{22}$ ✓ $\quad j=3$: $v_3 = v_3 - r_{23} q_2$ ... ... $v_n = v_n - r_{2n} q_2$
$i=3$: $q_3 = v_3 / r_{33}$ ✓

Save work! "Modified G-S"
demo: more stable: r_{ii} good down to 10^{-15} but Q not unitary.

— break (50 mins)

Task: Householder reflections.

$A =$ 

$Q_1 A =$ 

writing: left mult. acts on each col. of A separately
kill entire: new col $\propto e_1$

geom: