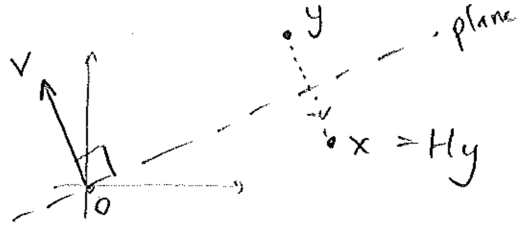
M116 WORKSHEET : Householder Reflectors

4/8/14.
Barnett.

We want unitary Q_1 turning $a_1 \in \mathbb{R}^m$ into $\begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Q_1 a_1$ $\quad (*)$

a) How big is α ?

Now find the matrix H which <u>reflects</u> $y \in \mathbb{R}^m$ through the plane through the origin with normal $v \in \mathbb{R}^m$:



Hints : use picture, try $\|v\| = 1$ case first.

b) You should get $H = I + (\text{rank-}1)$ What minimum effort is needed to apply this to a general vector y, ie the matvec $x = Hy$?

We pick Q_1 of the form H :

c) Explain why if $(*)$ holds then v must be of the form $\gamma\left(a_1 + \begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$ & we can choose $\gamma = 1$.

d) Let $f = \begin{bmatrix} a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, ie $a_1 = \begin{bmatrix} a_{11} \\ f \end{bmatrix}$ and so $v = \begin{bmatrix} a_{11} + \beta \\ f \end{bmatrix}$.

By considering $(*)$ & your formula for H, solve for β & simplify:

BONUS: which of the two solutions has less relative error due to rounding error ?

4/8/14. Barnett

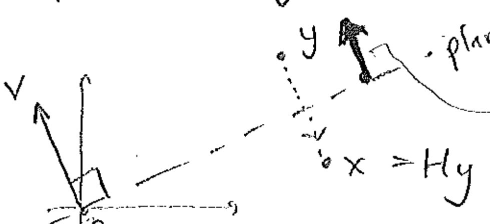
M116 WORKSHEET : Householder Reflectors

— SOLUTIONS —

We want unitary Q_1 turning $a_1 \in \mathbb{R}^m$ into $\begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Q_1 a_1$ (*)

a) How big is α ? $\alpha = \|a_1\|$ since $\|Q_1 a_1\| = \|a_1\|$ by unitarity.

Now find the matrix H which <u>reflects</u> $y \in \mathbb{R}^m$ through the plane through the origin with normal $v \in \mathbb{R}^m$:



Hints: use picture, try $\|v\|=1$ case first.

This vector $= (v^T y)\, v$ in the case $\|v\| = 1$.

$$x = y - 2\,(\text{this vector}) = y - 2v(v^T y)$$

Matrix doing this for general y is $H = I - 2vv^T$ (check it)

If $\|v\| \neq 1$ then $H = I - \underbrace{2\frac{vv^T}{v^Tv}}_{\text{indeed: (scalar)}vv^T}$

b) You should get $H = I + (\text{rank-1})$ What minimum effort is needed to apply this to a general vector y, ie the matvec $x = Hy$?

$$Hy = \underset{O(m) \text{ subtractions}}{y -} \underbrace{\frac{2}{v^Tv}(v^Ty)v}_{O(m)} \qquad O(m) \text{ effort.}$$

We pick Q_1 of the form H:

c) Explain why if (*) holds then v must be of the form $\gamma\left(a_1 + \begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right)$ & we can choose $\gamma = 1$. Since Ha_1 must be $\begin{pmatrix} * \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ then $\begin{bmatrix} v_2 \\ \vdots \\ v_m \end{bmatrix}$ must be $\propto \begin{pmatrix} a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$

d) Let $f = \begin{bmatrix} a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$, ie $a_1 = \begin{bmatrix} a_{11} \\ f \end{bmatrix}$ and so $v = \begin{bmatrix} a_{11} + \beta \\ f \end{bmatrix}$.

By considering (*) & your formula for H, solve for β & simplify:

$$\begin{bmatrix} \alpha \\ 0 \end{bmatrix} = Ha_1 = a_1 - 2\frac{v^Ta_1}{v^Tv}v = \begin{bmatrix} a_{11} \\ f \end{bmatrix} - 2\frac{v^Ta_1}{v^Tv}\begin{bmatrix} a_{11} + \beta \\ f \end{bmatrix}$$

since f's cancel, we must have $2\frac{v^Ta_1}{v^Tv} = 1$.

So, $v^Tv = 2v^Ta_1$, ie $a_{11}^2 + 2\beta a_{11} + \beta^2 + \|f\|^2 = 2(a_{11}^2 + \beta a_{11} + \|f\|^2)$ ie $\beta^2 = a_{11}^2 + \|f\|^2 = \|a_1\|^2$, $\beta = \pm\|a_1\|$

BONUS: which of the two solutions has less relative error due to rounding error? $\operatorname{sgn}(a_{11})$.