~~11.4~~6 Lec 15- 11/12/08. (1)

NSF DAY AT DARTMOUTH

September 11, 2008

MPS for EVPs

find: eigenpairs (E, u) for nontrivial u, $\begin{cases} -\Delta u = Eu & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega. \end{cases}$

Pick a freq. E.

- Basis rep. $u = \sum \alpha_n \phi_n(x)$ ← set of basis funcs. $(\Delta + E)\phi_n = 0$ in Ω. $\forall n$

bdry norm $\|u\|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega} |u(y)|^2 ds_y$

$$= \sum_{nm} \alpha_n \bar{\alpha}_m \underbrace{\int_{\partial\Omega} \phi_n(y)\overline{\phi_m(y)}\, ds_y}_{\text{matrix } F_{mn}\text{, Hermitian}} = \vec{\alpha}^* F \vec{\alpha}$$

called: quadratic form. note $\alpha^* F \alpha = 1$ is an ellipsoid

interior norm $\|u\|^2_{L^2(\Omega)} = \vec{\alpha}^* G \vec{\alpha}$, $G_{mn} = \int_\Omega \phi_n(x)\overline{\phi_m(x)}\, dx$ Gram matrix.

$\alpha G \alpha = 1$ another ellipsoid.

- If u sat. $(\Delta+E)u = 0$ in Ω,

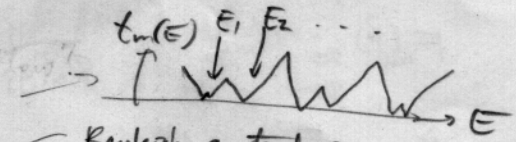
$t[u] = \dfrac{\|u\|_{\partial\Omega}}{\|u\|_\Omega}$ measures how 'close' u is to eigenfunc of Laplacian. Why? $t[u] = 0 \Leftrightarrow$ u is eigfunc. & E eigval

Rmk: also will see $t[u]$ small. $\Leftrightarrow$ E close to eigval.

- Method: i) guess E

ii) Find $\min\limits_{u \in \mathrm{Span}\{\phi_n\}} t[u] =: t_m(E)$

iii) search along E axis for minima where $t_m(E) \simeq 0$.



How do ii) ? $t_m^2(E) = \min\limits_{0 \neq \vec{\alpha} \in \mathbb{C}^N} \dfrac{\vec{\alpha}^* F \vec{\alpha}}{\vec{\alpha}^* G \vec{\alpha}}$ ← Rayleigh quotient for matrices (F, G)

$$= \min_{\alpha^* G \alpha = 1} \alpha^* F \alpha$$

Thm.: $\begin{cases} \text{Let } G \text{ be positive definite Hermitian, } F \text{ Hermitian, both } N\times N. \\ \text{Min. value of Rayleigh quot. } (F, G) \text{ is the minimum generalized eigenvalue } \mu_1 \text{ of } Fv = \mu G v \text{ (GEP)} \\ \text{and is achieved at } \alpha = v_1 \text{ the corresp. eigvec.} \end{cases}$

- i) Lemma: spectral thm for pair (F, G): GEP has complete set $\{v_i\}_{i=1}^N$ eigenvectors which sat. $v_i^* G v_j = \delta_{ij}$, ie G-orthog.

Remark: GEP common in structural engineering.

pf: spectral thm. for G: $G = W \Lambda W^*$ W unitary, Λ diagonal, (positive entries)

coordinate change $x = \underbrace{\Lambda^{1/2} W^*}_{\text{invertible}} v$ ie $v = W \Lambda^{-1/2} x$ $x, v \in \mathbb{C}^N$.

So $\underset{\text{(GEP)}}{Fv = \mu G v} \Leftrightarrow F W \Lambda^{-1/2} x = \mu \underbrace{G W \Lambda^{-1/2}}_{\text{cancels.}} x \Leftrightarrow \underbrace{\Lambda^{-1/2} W^* F W \Lambda^{-1/2}}_{\text{Hermitian since } F \text{ is.}} x = \mu x$

By spectral thm on $\Lambda^{1/2} W^* F W \Lambda^{-1/2} = X D X^*$ for some unitary $X = \begin{bmatrix} x_1 & \cdots & x_N \end{bmatrix}$ complete set eigvecs, o.n.b. for $\mathbb{C}^N$, D diagonal $= \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_N \end{bmatrix}$ real.

Then $v_i = \underbrace{W\Lambda^{-1/2}}_{\text{inv.}} x_i \quad i = 1 \ldots N$ are eigvecs of GEP, basis for $\mathbb{C}^N$

$v_i^* G v_j = x_i \underbrace{\Lambda^{-1/2} W^* G W \Lambda^{-1/2}}_{I} x_j = \delta_{ij}$ QED.

Now pf thm:

any α can be written $\sum_{i=1}^N \beta_i v_i$ since v_i basis.

$$F\alpha = \sum \beta_i \mu_i G v_i$$

$$\alpha^* F \alpha = \sum_{ij} \bar{\beta}_j \beta_i \mu_i \underbrace{v_j^* G v_i}_{\delta_{ij}} = \sum_i \mu_i |\beta_i|^2$$

$$\alpha^* G \alpha = \sum_i \bar{\beta}_j \beta_i v_j^* G v_i = \sum_i |\beta_i|^2$$

so $$\frac{\alpha^* F \alpha}{\alpha^* G \alpha} = \frac{\sum \mu_i |\beta_i|^2}{\sum |\beta_i|^2}$$ minimized by choosing $\beta_i = 0$ for $i \neq 1 \ldots N$ (obvious or prove it).

G's ellipsoid becomes sphere. F's aligned w/ axes.

May also prove via Lagrange multipliers.

In practice, we find as N incr, ϕ_n v. close to lin. dep.

ie nontriv. lin. comb of ϕ_n exp. small in Ω & on $\partial\Omega$.

$\Rightarrow$ F, G acquire common numerical nullspace, not full rank. (in floating pt. arith).

recipe (based on Fix & Heiberger, 1972, Vergini–Saraceno 1994) = code I gave you.

diag. $G \to W, \Lambda$ keep only λ's $> \underbrace{10^{-14}}_{\text{a regularizing parameter.}} \lambda_{max}$, put in $\tilde{\Lambda}$, kill corresp. cols of W.

$N \square$ $\leq N$ $\tilde{W}$ may now be rect.

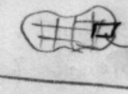
diag. $\tilde{\Lambda}^{-1/2} \tilde{W}^* F \tilde{W} \tilde{\Lambda}^{-1/2} \to X, D$

- How fill F, G matrices? naively each needs $O(N^2)$ integrals, each w/ quadrature.

$$F_{mn} \underset{\text{quadr.}}{\approx} \sum_{j=1}^M w_j \overline{\phi_m(y_j)} \phi_n(y_j) = \sum_j \bar{A}_{mj} A_{jn}$$ where $A_{jn} = \sqrt{w_j}\phi_n(y_j)$ last time.

ie $F = A^* A$, the 'square' of A.

sim $G = B^* B$ $\quad B_{jn} = \sqrt{w_j^I}\, \phi_n(z_j)$ $\quad z_j$ interior pts $j = 1 \ldots I$

 $w_j^I = \frac{\text{Vol}\,\Omega}{I}$. may be v. crude approx

- Note $t_m(E) = \min_{\alpha \neq 0} \sqrt{\frac{\alpha^* F \alpha}{\alpha^* G \alpha}} = \min_{\alpha \neq 0} \frac{\|A\alpha\|}{\|B\alpha\|}$ = min generalized sing. val. (A, B), GSVD.

Betcke's papers use this ... since no squaring, is actually more accurate.

- We need convergence in M & N: usually $M > N$ but same order, eg. $2N$. got to here.

Error analysis: if $t[u]$ small, how close is E to eigval E_j of domain?

Thm: (Moler–Payne, 1967) ~~Still, 1989~~ Kuttler–Sigillito '78

Let $(\Delta + E)u = 0$ in Ω,

then $\exists j$ st. $\frac{|E - E_j|}{E_j} \leq C_\Omega t[u]$ const dep. only on domain Ω

relative error in eigenvalue

means.  smaller t $\Rightarrow$ E closer to E_j

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State thm on WS. , give example of domain: $A = \frac{d^2}{dx^2}$ in $[0,1]$ may act on funcs. in C^2.
* self-adjointness of Δ in Ω only w/ ~~zero~~ homog. BCs. (p. ④) 11/18).
→ do WS.

eg. $A = -\Delta$ with $\mathcal{D}(A) = \{u : u \in C^2(\Omega) \cap C(\bar{\Omega}),\ u|_{\partial\Omega} = 0\}$

Let $\tilde{A}$ be extension of operator A to larger domain, ie $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A}) \subset \mathcal{H}$

we will apply to $\tilde{A} = -\Delta$ on $L^2(\Omega)$ with no BCs. and $\tilde{A} = A$ in $\mathcal{D}(A)$

prove Thm. Let $\begin{cases} u \in \mathcal{D}(\tilde{A}) \\ \tilde{A}u = Eu \end{cases}$, and w be st. $\begin{cases} u - w \in \mathcal{D}(A) \\ \tilde{A}w = 0 \quad (+) \end{cases}$, ie w has same bdry values as u.

ie u sat Helm. in Ω, but not nec 0 on bdry. ← w harm. in Ω.

$$u - w = \sum c_i \phi_i$$

$$\sum c_i E_i \phi_i = A(u-w) \overset{\text{extension}}{=} \tilde{A}(u-w) \overset{(+)}{=} Eu \qquad (*)$$

and. $$\sum c_i^2 (E - E_i)^2 = \| \underbrace{\sum c_i (E - E_i)\phi_i}_{\text{since } E(u-w) - \tilde{A}(u-w) = (\tilde{A} - E)w = -Ew} \|^2 = E^2 \|w\|^2$$

$$\| $$

$$\sum c_i^2 E_i^2 \frac{(E-E_i)^2}{E_i^2} \geq \min_i \left(\frac{E - E_i}{E_i}\right)^2 \cdot \underbrace{\sum c_i^2 E_i^2}_{E^2 \|u\|^2 \text{ since } (*)}$$

so $$\min_i \frac{|E - E_i|}{E_i} \leq \frac{\|w\|}{\|u\|}$$

A-posteriori estimate.

Finally $\|w\|_\Omega \leq C_\Omega |w|_{\partial\Omega} = \|u\|_{\partial\Omega}$.

(Similar bound on eigenmode.)

Use for error bounds on dist. to eigenvalue.

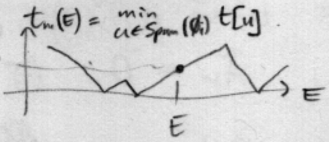
Error anal. of MPS for eigenmodes: ϕ_j & eigvals E_j of domain Ω.

Thm (Moler-Payne '67, Kuttler-Sigillito '78) Let $(\Delta+E)u=0$ in Ω, bdry err. $t[u] := \frac{\|u\|_{\partial\Omega}}{\|u\|_{\Omega}}$

then $\exists j$ s.t. $\frac{|E-E_j|}{E_j} \le C_\Omega\, t[u]$

rel. err. in eigval. — const. dep only on domain, $O(1)$.

Usage: MPS curves $t_m(E) = \min_{u\in \mathrm{Span}(\phi)} t[u]$



then smaller $t[u]$ is, e.g. 10^{-6}, closer there must be a true eigval. E_j. e.g. 10^{-6} rel. err.

Domain of op. A =: $\mathcal{D}(A) \subset \mathcal{H}$ ← Hilbert space: complete lin. space (Banach) w/ 2-norm.

e.g. $\mathcal{H} = L^2(\Omega)$, $A = -\Delta$, may define $\mathcal{D}(A) = \{u \in C^2(\Omega) \cap C(\bar\Omega),\ u|_{\partial\Omega} = 0\}$ so 2nd derivs cont.; ie also cont. on $\partial\Omega$; vanish on bdry

note $\mathcal{D}(A) \ne \mathcal{H}$ since A not bounded, so can't act on every element of $\mathcal{H}$ (since $\mathcal{H}$ complete, contains its limit points, is closed).

However, this $\mathcal{D}(A)$ is dense in $\mathcal{H}$: you can get arb. close to every $f \in L^2(\Omega)$ with cont. fun van. on $\partial\Omega$

Such $u \in \mathcal{D}(A)$ are 'classical solns' to PDE; may expand to a Sobolev space for weak solns.

We assume A has point spectrum, ie countable set of eigvals E_j, whose eigfuncs are complete o.n.b. for $\mathcal{H}$

sufficient condition is: A^{-1} cpt, true for diff. ops since kernel of A^{-1} is Greens func, continuous or weakly singular

Thm: define residual $r := Au - Eu$, for $u \in \mathcal{D}(A)$.

then i) $\exists j$ s.t. $|E - E_j| \le \frac{\|r\|}{\|u\|}$

WS.

note: $\|r\|$ is size of failure of

similarly may prove ii) $\exists$ eigenfunc ϕ_j s.t. $\frac{\|u-\phi_j\|}{\|u\|} \le \varepsilon\sqrt{1+\varepsilon^2}$, where $\varepsilon := \frac{\|r\|}{\|u\|\, d_j(E)}$

$d_j(E) := \min_{E_k \ne E_j} |E_k - E|$ dist to next nearest eigval.

Application: for above A, $\mathcal{D}(A)$, say $\{E, u\}$ trial eigenval, mode produced by some method (eg FEM) with correct (homog.) BCs but not satisfying PDE,

ie $(\Delta + E)u = -r$, $r \ne 0$, then $\|r\|$ bounds dist. to nearest true E_j & norm error in u from ϕ_j.

'a-posteriori' err. bnd, common in num. anal.

Pron Moler-Payne:

choose A, $\mathcal{D}(A)$ as above, and $\tilde{A}$ an extension of A to larger domain $\mathcal{D}(A) \subset \mathcal{D}(\tilde{A}) \subset \mathcal{H}$.

this means $\tilde{A}\big|_{\mathcal{D}(A)} = A$, ie $\tilde{A}u = Au \quad \forall u \in \mathcal{D}(A)$.

choose $\mathcal{D}(\tilde{A}) = C^2(\Omega) \cap C(\bar\Omega)$ with no BCs on u.

Approximation theory for MPS for Laplace BVP:

Ω = ~~bounded domain~~ ; ~~analytic~~ simply-connected, in $\mathbb{R}^2$.

Basis funcs $\phi_n(r,\theta) = e^{in\theta} r^{|n|}$
regular sols to $\Delta\phi_n = 0$ in $\mathbb{R}^2$

Thm: ϕ_n are dense in the space of solutions to $\Delta u = 0$, in $L^2(\Omega)$ norm.

Ie, for any soln. $\Delta u = 0$ in Ω,

$$\min_{\vec{c} \in \mathbb{C}^{2N+1}} \Big\| \sum_{|n| \le N} c_n \phi_n - u \Big\|_{L^2(\Omega)} \to 0 \quad \text{as } N \to \infty.$$

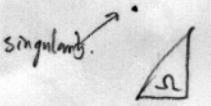
proofs: Gaier book.

- Same result in sup norm.
- Example of Runge's Thm (1885): approximation of any complex analytic func on Ω by poly's $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots$
  Weierstrass is special case of $\Omega = [a,b]$ on real axis.
- connection to analytic funcs: every harm. $u(x_1,x_2)$ may be written as $\mathrm{Re}[g]$ for $g(z)$ analytic. $z = x_1 + ix_2$
  $\phi_n - \phi_{-n}$, $\phi_n + \phi_{-n}$ are then Re, Im parts of z^n, ie complex polynomials.
- if Ω not simply-conn, need basis funcs to include singularities in each conn. component of $\mathbb{R}^2 \setminus \bar{\Omega}$.

If soln. u may be analytically continued beyond $\bar{\Omega}$ a finite distance,
eg. if f analytic on $\partial\Omega$, Ω analytic bdry.

then $\exists \rho > 1$ s.t. $\displaystyle \min_{\vec{c} \in \mathbb{C}^{2N+1}} \Big\| \sum_{|n| \le N} c_n \phi_n - u \Big\|_{L^\infty(\Omega)} = O(R^{-N})$ for every $R < \rho$, but no $R > \rho$.

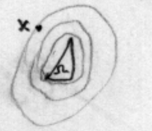
Furthermore, ρ is "conformal dist." of nearest singularity of u to $\partial\Omega$.



Defn of conf. dist $\rho(\vec{x})$ for any $\vec{x} \in \mathbb{R}^2 \setminus \bar{\Omega}$: solve
$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^2 \setminus \bar{\Omega} \\ v = 0 & \text{on } \partial\Omega \\ v - \ln|x| = O(1). \end{cases}$$

then $\rho(\vec{x}) := e^{v(\vec{x})}$

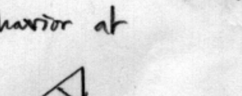
ext BVP w/ soln. asymp. to $\ln r$.



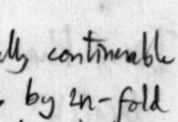
- note how similar to {interpolation / quadrature} convergence rates.

Through theory of Vekua, (see Betcke thesis, Henrici review), convergence rates carry over to 2nd order elliptic PDE w/ analytic coeffs, eg $(\Delta + E)u = 0$ Helmholtz. Eg HW7 #1: singularity gives conv. rate of ...

EVP behavior at Corners:



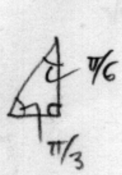
$\beta = \frac{\pi}{n}$ $n \in \mathbb{Z}$ called regular: modes ϕ_j analytically continuable beyond corner, by 2n-fold reflection.



Otherwise singular.

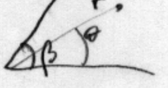
eg.

Note HW7



all regular ⇒ eigenmode analytically continued, in fact to $\mathbb{R}^2$ since tiling the plane.

singular corners:



to regain exponential convergence need $J_\nu(kr)\sin\nu\theta$ for $\nu = \frac{\pi}{\beta} n$
fractional-order Bessels, $n \in \mathbb{N}$.

show mushroom pics?