(Green Rep. Formula). Let u be a Helmholtz soln (of radiating type for exterior case), then

for $x \in \left\{ \begin{array}{ll} \Omega & \text{(interior)} \\ \mathbb{R}^d \setminus \overline{\Omega} & \text{(exterior)} \end{array} \right.$ $\quad u(x) = \pm \int_{\partial\Omega} \left[u_n(y) \Phi(x,y) - u(y) \frac{\partial \Phi(x,y)}{\partial n_y} \right] ds_y$

(+) interior
(−) exterior

pf: • interior pf: same as GRF for Laplace op, since singularity same.
• exterior uses radi. cond. (CK Thm 2.4).

BIE soln. for u^s: $u^s = D\tau$ sat Helmh. in $\mathbb{R}^d \setminus \overline{\Omega}$, radiative.

solves exterior BVP if $\underbrace{(I + 2D)\tau}_{2u^{s+} \text{ by JR3}} = 2f = -2u^i|_{\partial\Omega}$ inc. field.

However, there's a problem: $(I + 2D)$ not injective for certain discrete set of k, so nonunique soln, numerically get cond # → ∞

Interpretation. Thm: say $k^2 = E_j^{(N)}$ for which $\left\{ \begin{array}{ll} -\Delta u = k^2 u & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega \end{array} \right\}$ has nontriv soln.

interior Neumann eigenvalues of Ω.

then. $\dim \text{Nul}(I + 2D) > 0$.

pf: SLP $u := S\sigma$, so nontriv $u \Rightarrow$ nontriv. σ. JR2 says $0 = u_n^- = (D^T + \frac{1}{2})\sigma$

$\Rightarrow \dim \text{Nul}(I + 2D^T) > 0$

By Fredholm alternative for cpt op D, $\dim \text{Nul}(I + 2D) > 0$.

— Show evolving eigvals of 2D as k increases... Matlab. dlp_eigvalsweep.m where hit -1 get Neumann eigvals, $+1$ " Dirichlet-

got to here.

Stop interior Neumann resonances from plaguing scattering soln: use rep. $u^s = (D - i\eta S)\tau$, $\eta > 0$

This solves ext. BVP if $\underbrace{(I + 2D - 2i\eta S)\tau}_{2u^{s+} \text{ by JR3,1}} = -2u^i|_{\partial\Omega}$.

Brakhage–Werner, Leis, Panich (196[?])

Thm: $\frac{1}{2}I + D - i\eta S$ injective $\forall k > 0$

pf: suppose $(\frac{1}{2} + D - i\eta S)\tau = 0$, wish to show $\tau \equiv 0$.

say $v := (D - i\eta S)\tau$, then $v^+ = 0$ by construction of BIE.

$\Rightarrow v = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$ by uniqueness of ext. Dir. BVP, for radiative solns

$\Rightarrow v_n^+ = 0$ on $\partial\Omega$.

$\left. \begin{array}{l} \text{JR1,3} \Rightarrow v^- = -\tau \\ \text{JR2,4} \Rightarrow v_n^- = -i\eta\tau \end{array} \right\} (a)$

GT1 in Ω gives $\underbrace{\int_{\partial\Omega} \overline{v^-} v_n^- \, ds}_{\text{from (a)}\ i\eta \int |\tau|^2 ds} = \underbrace{\int_\Omega \overline{v} \Delta v + \overline{\nabla v} \cdot \nabla v \, dx}_{\int_\Omega -k|v|^2 + |\nabla v|^2 dx \text{ pure real.}}$

since $\eta \neq 0$, take Im part shows $\tau \equiv 0$, QED.

demos · ~~Dirichlet eigfuncs. of Ω.~~ ~~Neumann " "~~ (5)

· In practice, choose $\eta \approx k$.

Note: $D - i\eta S$ has weakly-singular kernel since $S(s,t) \simeq \frac{1}{2\pi} \ln |y(s) - y(t)| + O(1)$

Need quadrature rule for singular integrand.
Several ways to handle this:

i) Martensen–Kussmaul (Kress 1991) CK books, give quadr weights, on uniform periodic ~~grid~~ nodes, for integration of $\ln\left(4\sin^2\frac{t-t_j}{2}\right) f(t)$, f anal, 2π-periodic, when t_j is one of the nodes. 2π periodic log sing.
They split $D(t,t_j) - i\eta S(t,t_j) = \ln\left(4\sin^2\frac{t-t_j}{2}\right) K_1(t,t_j) + K_2(t,t_j)$ for analytic K_1, K_2 ⇒ spectral convergence, (excellent) but work needed to derive weights & split.

ii) Kapur–Rokhlin (1997) quadr weights for unif periodic nodes, which can handle smooth + $(\ln|t-s|)$ smooth.
· diagonal values $D(t_i,t_i)$ or $S(t_i,t_i)$ ($=\infty$) not needed.
· algebraic conv. up to ~10th order, but prefactor bad

iii) Alpert (1999) hybrid Gauss–trapezoidal: use non-equispaced nodes near singularity. ← nice projects ↑

· In v. large systems ($N > 10^4$) it becomes better to use Fast Multipole Method to compute action of D or S on density τ, a fast matrix-vector product in $O(N\ln N)$ flops.
The lin. sys. is then solved via iterative meth. which require only $x \mapsto Ax$ products (a few tens of them).

FMM key idea: field due to arb. # sources inside ball can be rep. outside ball by fixed # L.I. funcs.

· Transmission probs: eg acoustic scatt off medium of diff wave speed.

medium, refractive index $n = \frac{k}{k_0}$ (speed n times slower).

$$\left.\begin{aligned} (\Delta + k_0^2)u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega} \\ (\Delta + k^2)u &= 0 \quad \text{in } \Omega \\ u &\text{ cont. on } \partial\Omega \\ u_n &\text{ cont. " "} \end{aligned}\right\}$$

we say $u = u^i + u^s$, $u^i = \begin{cases} e^{ik_0 k\cdot x} & \text{outside} \\ 0 & \text{inside} \end{cases}$

So u^s obeys PDE
$u^{s+} - u^{s-} = u^{i-} - u^{i+}$
$u_n^{s+} - u_n^{s-} = u_n^{i-} - u_n^{i+}$ given RHS funcs. f, g.

Say $\left.\begin{aligned} u^s &= S_0\sigma_0 \quad \text{outside} \\ u^s &= S\sigma \quad \text{inside,} \end{aligned}\right\}$ densities indep, σ, σ_0 unknowns $\Bigg\}$ then $S_0\sigma_0 - S\sigma = f$

$(D_0^T - \tfrac{1}{2})\sigma_0 - (D^T + \tfrac{1}{2})\sigma = g$.

$$\text{ie} \quad \begin{bmatrix} S_0 & -S \\ D_0^T + \frac12 & -D^T + \frac12 \end{bmatrix} \begin{bmatrix} \sigma_0 \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

coupled BIEs, but not 2nd kind since not of form $\left(\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{pmatrix} K_1 & K_2 \\ K_3 & K_4 \end{pmatrix} \right) \begin{bmatrix} \sigma_0 \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$.

Correct way is use $u^s = D_0\tau + S_0\sigma$ outside
$u^s = D\tau + S\sigma$ inside.

which gives 2nd kind & cancels singularities. trick of (Rokhlin 1983)

$$\underbrace{\begin{bmatrix} D_0 + \frac12 - (D - \frac12) & S_0 - S \\ T_0 - T & D_0^T + \frac12 - (D^T - \frac12) \end{bmatrix}}_{\to\ I\ +\ \text{compact.}} \begin{bmatrix} \tau \\ \sigma \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

use Nystrom to get τ, σ then recon. u^s field.

M116. Lec 14. (1)

refs: my MFS paper w/ Timo.

· debug the ...
· exterior rec. prob BWLP.
MPS
→ complete Bessel sets in $\mathbb{R}^2$
least-sq. BVP.
EVP.

Debugging: i) bug ≈ needle in haystack
→ break haystack into smallest poss pieces, test them as you go.
Eg. defining domain w/ y_j bdry pts, n_j normals, plot it before you move on.

ii) examine visually everything you can.

Recall BIE: $(I+2D)\tau = 2f$ for exterior radiative BVP u^s, but $2D$ has eigval -1 whenever k is interior Neumann eigval
causes cond$(I+2D)\to\infty$ at such k, even though exterior u^s doesn't blow up ⇒ numerical problem.
Contrast: interior BVP for generic f, does blow up (ie physical pole) (most of you tested on f from BVP which doesn't — subtle).

→ BWLP. (last time 4)
(skip 5).

MPS. approx $u(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$. basis funcs, particular solns to PDE, but not the BCs.
Then adjust $\{\alpha_j\}_{j=1}^{N} =: \vec{\alpha}$ coeff. vector to get BCs close to correct.

eg. Laplace $\Delta u = 0$ in Ω, $u = f$ on $\partial\Omega$.
Recall complex anal: $f(z)$ anal at 0, has radius of conv. $r>0$
meaning Taylor $f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \ldots$ conv in disc $|z|<r$, diverge outside $|z|>r$.

coeffs fixed by beh. at 0.

unfinished.

Skipped.

2) Thm (Faber, Szegő, Walsh):
if f anal in open $D \supset \bar{\Omega}$, $\exists$ seq. polys $p_n(z)$ $n = 0,1,2,\ldots$
s.t. $|f(z) - p_n(z)| \leq C K^{-n}$
for some $K>1$
(in fact K is 'conformal dist' to nearest sing)

The p_n polys will not in general be Taylor series, coeffs change each time

is every harm u the Re of analytic func?

the contrary. Similarly, a referee's report is normally provided with the understanding that the name of the writer be withheld from certain interested parties and the referee must be anonymous unless otherwise indicated in advance. The writer of the recommendation or report must respond fully and keep confidential any privileged information, personal or mathematical, that the writer receives. If the requesting individual, institution, agency or company becomes aware that confidentiality or anonymity cannot be maintained, that should be immediately communicated.

Where choices must be made and conflicts are unavoidable, as with editors or those who decide on appointments or promotions, it is essential to keep careful records that would demonstrate the process was fair ...

Freedom to publish must sometimes yield to security concerns, but mathematicians should resist excessive secrecy demands whether by government or private institutions.

When mathematical work may affect the public health, safety or general welfare, it is the responsibility of mathematicians to disclose the implications of their work to their employers and to the public, if necessary. Should this bring retaliation, the Society will consider what action it may take to help the "whistle-blower", particularly when the disclosure has been made to the Society.

No one should be exploited by the offer of a temporary position at an unreasonably low salary and/or an unreasonably heavy work load.

III. EDUCATION AND GRANTING OF DEGREES

Holding a Ph.D. degree is virtually indispensable to an academic career in mathematics and is becoming increasingly important as a certificate of competence in the wider job market. An institution granting a degree in mathematics is certifying that competence and must take full responsibility for it by insuring the high level and originality of the Ph.D. dissertation work, and sufficient knowledge by the recipient of important branches of mathematics outside the scope of the thesis. When there is evidence of plagiarism it must be carefully investigated, even if it comes to light after granting the degree, and, if proven, the degree should be revoked.

Mathematicians and organizations involved in advising graduate students should fully inform them about the employment prospects they may face upon completion of their degrees.

②

MPS for interior BVP: $\begin{cases}(\Delta+k^2)u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega\end{cases}$

Laplace $k=0$: $\phi_n(x) = e^{in\theta} r^{|n|}$, $n\in\mathbb{Z}$

$k>0$: $\phi_n(x) = e^{in\theta} J_n(kr)$ (complete for regular solutions in the plane).

Choose truncation $-N \le n \le N$, ie $2N+1$ basis funcs.

Given $\vec{\alpha} \in \mathbb{C}^{2N+1}$, $u^{(N)}(x) = \sum_{n=-N}^{N} \alpha_n \phi_n(x)$ is interior PDE soln

BC error func is $u^{(N)}\big|_{\partial\Omega} - f$, want as small as poss.

$\Rightarrow \min_{\vec{\alpha}} \int_{\partial\Omega} \left|\sum_n \alpha_n \phi_n(x) - f(x)\right|^2 ds_x$ — square of $L^2(\partial\Omega)$ norm.

quadrature nodes $y_j \in \partial\Omega$, $j=1\ldots M$, weights w_j: $\int_{\partial\Omega} g(y)\, ds_y \approx \sum_{j=1}^{M} w_j g(y_j)$

$\sum_j w_j \left| \sum_n \alpha_n \phi_n(y_j) - f(y_j)\right|^2$

$= \left\|\sum_n A_{jn}\alpha_n - b_j\right\|_2^2$ ← Euclidean norm in $\mathbb{C}^M$, where $A_{jn} := \sqrt{w_j}\,\phi_n(y_j)$ rect. matrix ($M \times 2N+1$)

$b_j := \sqrt{w_j}\, f(y_j)$ rhs column vector.

For what $\vec{\alpha}$ is $\|A\vec{\alpha} - \vec{b}\|^2$ minimized? linear least squares problem (lin. alg.), usually overdetermined (A 'tall', ie more $M > \underbrace{2N+1}_{\#\text{ basis funcs}}$)

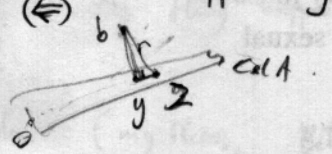
you may solve by Matlab's \ command.

↳ What is actually going on here? $r = A\vec{\alpha} - \vec{b}$ resid. vec.

Thm: a vector $\vec{\alpha}_0$ minimizes norm $\|r\|_2$ iff $r \perp \text{Col}\,A$, ie $A^* r = 0$

(mult. by A^*) ie $A^*A\alpha = A^*b$ (Normal Eqns).

Pf: ($\Leftarrow$): suppose y is pt in Col A ~~...~~ st. $r \perp \text{Col}\,A$.



z is another pt $\in$ Col A: $\|b-z\|^2 = \|b-y\|^2 + \|y-z\|^2 \geq \|r\|^2$

by Pythagoras. $\|b-y\|^2 = \|r\|^2$

so no other residual is smaller.

Solve Normal eqns. by SVD $A = U\Sigma V^*$ $\quad A^* = V\Sigma^* U^*$

$A^*A = V\Sigma^2 V^*$

so $V\Sigma^*\Sigma V^*\alpha = V\Sigma^* U^* b$.

V invertible, Σ^* has no zero entries if A full rank $\Rightarrow \Sigma V^*\alpha = U^* b$

ie $\alpha = \underbrace{V\Sigma^{-1}U^*}_{A^+}b$. A^+, pseudoinverse

If A not full rank, Σ^{-1} is used with ~~...~~

least-sq algorithms are backwards-stable: solves exactly for some $A + \delta A$ with $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_{mach})$.

Summary: set up matrix A whose cols. are the basis funcs eval. at bdry nodes, ③

(rhs. vector

solve $A\vec{\alpha} = \vec{b}$ in least-sq. sense.

Evaluate soln. $u^{(N)} = \sum \alpha_n \phi_n(x)$ at whichever $x \in \Omega$ you need.

Eigval probs: find E st. $\exists$ nontriv u st $\left\{ \begin{array}{ll} -\Delta u = Eu & \text{in } \Omega. \\ u = 0 & \text{on } \partial\Omega \end{array} \right.$ ⟵ Helm $(\Delta + k^2)u = 0$ with eigval $E = k^2$

Dirichlet.

MPS: naively: want non-triv $\vec{\alpha}$ st. $u^{(N)} = \sum \alpha_n \phi_n(x)$ vanishes on $\partial\Omega$

recall basis funcs. dep. on k.

if such $\vec{\alpha}$ exists, k^2 is an eigval E_j & $u^{(N)}$ is corresp. efunc u_j.

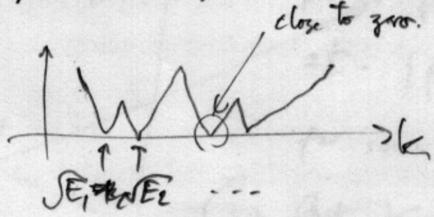
meas. vanishing by $\|u^{(N)}\|^2_{L^2(\partial\Omega)} \overset{\text{quad}}{\approx} \sum w_j \left|\phi_n(y_j)\alpha_n\right|^2 = \|A\vec{\alpha}\|_2^2$

$\int_{\partial\Omega} \left|\sum \alpha_n \phi_n(y)\right|^2 ds_y$

we minimize $|A\alpha|$ while holding $\|\alpha\| = 1$ to $\overset{\text{try } k}{\text{prevent}}$. triv. solns.

but $\min_\alpha \frac{\|A\alpha\|}{\|\alpha\|} = \sigma_1$ min. sing. val of A.

Now sweep k, plotting $\sigma_1 = \sigma_1(k)$:



if N chosen carefully.

got to here.

However to increase accuracy of rep. of eigenfuncs, want incr. N.

Interesting thing happens:

useless: cannot find minima.

as N incr, There exist $u^{(N)} = \sum \alpha_n \phi_n(x)$ with $|\alpha| = 1$ which are exp. small in Ω, ie $\|u\|_{L^2(\Omega)} \to 0$

Eg: Bessel funcs behave like $J_n(z) = \frac{1}{n!}\left(\frac{z}{2}\right)^n \in O(z^n)$ $\to 0$ as $n \to \infty$ for z fixed.

Soln. (my thesis, Betcke–Trefethen 2005):

$\min_\alpha \frac{\|u\|_{L^2(\partial\Omega)}}{\|u\|_{L^2(\Omega)}}$

interior norm: $\|u\|^2_{L^2(\Omega)} = \int_\Omega |u|^2 dx \approx \frac{1}{I}\sum_{j=1}^{I} |u(z_j)|^2$.

can approx. quite crudely via I int. pts. z_j