MATH 116 WORKSHEET : Laplace's eqn.

10/28/08
Barnett.

GT1) Prove $$\int_\Omega (u\Delta v + \vec{\nabla} u \cdot \vec{\nabla} v)\, dx = \int_{\partial\Omega} u\, \hat{n}\cdot\vec{\nabla} v\, ds$$

[Hint: what $\vec{a}$ in Divergence Thm. has RHS as outgoing flux?]

GT2) Prove $$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} (u v_n - v u_n)\, ds$$

- Finally, complete proof that $\Delta_x \ln|x| = 0$ for $x \neq 0$:

$x = (x_1, x_2)$

$$\frac{\partial}{\partial x_1} \ln|x| = \frac{1}{2}\frac{\partial}{\partial x_1} \ln(x_1^2 + x_2^2) = \frac{1}{2(x_1^2+x_2^2)}\cdot 2x_1 = \frac{x_1}{|x|^2}$$

$$\frac{\partial^2}{\partial x_1^2} \ln|x| = \cdots$$

$$\Delta \ln|x| = \cdots$$

MATH 116 WORKSHEET : Laplace's eqn.

10/29/08
Barnett.

~ SOLUTIONS ~

(GT1) Prove $\int_\Omega (u\Delta v + \vec\nabla u\cdot\vec\nabla v)\,dx = \int_{\partial\Omega} u\,\underbrace{\hat n\cdot\vec\nabla v}\,ds$

[Hint: what $\vec a$ in Divergence thm. has RHS as outgoing flux?]

$$\vec a = u\vec\nabla v \qquad \text{check} \qquad \hat n\cdot\vec a = \vec n\cdot(u\vec\nabla v) = u v_n \ \checkmark$$

Div. Thm:

$$\int_\Omega \vec\nabla\cdot\vec a\, dx = \int_{\partial\Omega} \hat n\cdot\vec a\, ds$$

$$\underbrace{\vec\nabla\cdot(u\vec\nabla v) = u\Delta v + \vec\nabla u\cdot\vec\nabla v}_{\text{prod rule for div.}} \qquad \text{QED.}$$

(GT2) Prove $\int_\Omega (u\Delta v - v\Delta u)\,dx = \int_{\partial\Omega} (u v_n - v u_n)\,ds$

$u \leftrightarrow v$ in GT1: $\int (v\Delta u + \vec\nabla u\cdot\vec\nabla v)\,dx = \int v u_n$.

subtract from GT1, $\vec\nabla u\cdot\vec\nabla v$ cancels, QED.

Finally, complete proof that $\Delta \ln|x| = 0$ for $x\neq 0$:

$x = (x_1, x_2)$ $\qquad \frac{\partial}{\partial x_1}\ln|x| = \frac12 \frac{\partial}{\partial x_1}\ln(x_1^2+x_2^2) = \frac{1}{2(x_1^2+x_2^2)}\cdot 2x_1 = \frac{x_1}{|x|^2}$ if $x\neq\vec 0$

$$\frac{\partial^2}{\partial x_1^2}\ln|x| = \frac{\partial}{\partial x_1}\left(\frac{x_1}{|x|^2}\right) = \frac{1}{|x|^2} + x_1 \underbrace{\frac{-1}{|x|^4}}_{\text{careful since } (x_1^2+x_2^2)^2}(2x_1) = \frac{1}{|x|^4}\left(|x|^2 - 2x_1^2\right)$$

$$\Delta \ln|x| = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right)\ln|x| = \frac{1}{|x|^4}\left(2|x|^2 - 2x_1^2 - 2x_2^2\right) = 0. \qquad \text{QED.}$$