Math 116 Numerical PDEs: Homework 3

due Fri midnight, Oct 17

Don't forget I am grading you both on coding (i.e. building, testing and debugging the algorithms), and on understanding (analysing, communicating, relating to lecture material, proving things). If you merely code, you cannot get a top score. This reflects the pleasing mixture that is the field of numerical analysis.

1. [Jon's question]

   (a) Plot the function $f(x,y) = (x^2+y^2)^{-1}\log(e^{x^2+y^2} - x^2 + 2)$ for $x, y \in [0.5, M]$ where M is large enough to see the asymptotic behavior of f as $x^2+y^2 \to \infty$. Give a 3D plot and a contour plot.

   (b) Plot the absolute value of $f(z) = (1+25z^2)^{-1}$ in the complex plane Re z, Im $z \in [-1,1]$ using the plot type of your choice to illustrate best the *simple poles* of f. Where are they? (show your algebra). Notice the connection to Runge's phenomenon.

   [Hint: `[xx, yy] = meshgrid...; zz = xx+1i*yy`]

2. Prove that, given a set of distinct points $\{x_j\}_{j=0,\ldots n}$ in $[a,b]$ there exists a *unique* set of weights $\{w_j\}_{j=0,\ldots,n}$ such that Newton-Cotes quadrature integrates exactly over $[a,b]$ all polynomials up to degree n. Use a different method than in lecture. [Hint: write the equations the weights must satisfy using the simplest basis for $\mathbb{P}_n$]

3. Consider numerical integration of $(1+x^2)^{-1}$ on $[-1,1]$. Produce a single log-linear plot comparing the convergence vs $n = 1,\ldots,40$ of the quadrature error using $n+1$ equally-spaced nodes with the following two schemes for weights:

   (a) Composite trapezoid rule

   (b) Newton-Cotes, *i.e.* interpolatory quadrature. You may use a linear solve to get the weights. [Hint: see previous question]

   Note that the exact integral is easy to compute. Then replot the composite trapezoid data on a more suitable plot to show its convergence as a straight line. Does it match the theorem from lecture? What is the minimum acheivable error in Newton-Cotes, and why is it not close to ε_{mach}? [Hint: it is not due to a Runge phenomenon since for this function the poles are sufficiently far away to stop that]

4. We discussed that given an inner product $(\cdot,\cdot)$ on $L^2[-1,1]$ a sequence (q_n) of orthogonal polynomials can be constructed by applying the Gram-Schmidt procedure from linear algebra to the monomials $1, x, x^2, \ldots$. Prove that the following 3-term recurrence relation also constructs them:

$$\begin{aligned} q_0(x) &= 0 \\ q_1(x) &= 1 \\ q_{j+1}(x) &= xq_j(x) - \alpha_{j+1}q_j(x) - \beta_{j+1}q_{j-1}(x), \qquad j = 1,2,\ldots \end{aligned}$$

   where $\alpha_{j+1} := (q_j, xq_j)/(q_j,q_j)$ and $\beta_{j+1} := (q_j,q_j)/(q_{j-1},q_{j-1})$. [Hint: one approach is to start with Gram-Schmidt and notice all but two projection terms vanish. Another is to prove that the above recurrence generates a mutually-orthogonal set]

5. Get from the website `gauss.m` which provides nodes and weights of n-node Gaussian quadrature on $[-1,1]$. Use this to produce convergence plots of quadrature error for the integrals of the following functions on $[-1,1]$. (If possible make a single figure with 4 labeled subplots in it) Note, in each case you can compute the analytic answer to compare to.

   (a) $(1+x^2)^{-1}$ (is performance better than in question #3?)

   (b) x^{20}

   (c) $|x|^3$

   Measure the order (if algebraic) or power law (if exponential) of convergence in each case, or state another kind of behavior, as appropriate. Discuss reasons for what you observe. [Hint: smoothness of the function]

   BONUS: In the last case, can you think of a way to handle this type of problem while preserving high convergence rate?

6. Produce a convergence plot of quadrature error for the periodic n-point trapezoid rule for the 2π-periodic entire function $(1/2\pi)e^{\cos x}$ on $[0,2\pi)$, for $n = 1,\ldots,20$. The exact answer is the modified Bessel function $I_0(1)$ which in Matlab is `besseli(0,1)`. Does the convergence appear algebraic, exponential, or super-exponential (*i.e.* log of error converges faster than linear) before ε_{mach} is reached?

7. Repeat the previous question for the function $\left(1+\cos^2(x/2)\right)^{-1}$ which is analytic on the real axis. You don't have an 'exact' answer to compare to (well, you do if want to attempt a nice substitution, but don't spend time on this...), so observe when convergence to ε_{mach} has occurred, and use that answer. Tell us what the answer is to 15 digits, and how many nodes were needed for this. Then apply the theorem on convergence rate for periodic analytic functions to predict the rate, and add this line to your plot. It should match.

8. Repeat question #6 for the C^∞ periodic function $\exp(-1/|\sin(x/2)|)$ which however is not analytic on the real line (it has essential singularities wherever its limit is zero). How does convergence now appear? Is it algebraic, or exponential, or ...? Don't be afraid to explore higher n to reach a solid conclusion. Please try to categorize the behavior.