Math 116 Numerical PDEs: Homework 1

due Fri midnight, Oct 3

Questions vary in difficulty and length. Post as much as you can to your webpage, for instance including all your codes, which should be concise with the occasional comment. The rest you may do on paper and leave at my office (or, ideally, use LaTeX and post on your webpage like I do with this homework).

1. If $\mathbf{u}$ and $\mathbf{v}$ are m-vectors, the matrix $A = I + \mathbf{u}\mathbf{v}^*$ is known as a *rank-one perturbation of the identity*. Show that if A is non-singular, then its inverse has the form $A^{-1} = I + \alpha\mathbf{u}\mathbf{v}^*$ for some scalar α, and give an expression for α. For what $\mathbf{u}$ and $\mathbf{v}$ is A singular? If it is singular, what is Nul A? [NLA Ex 2.6]

2. The spectral radius $\rho(A)$ of a square matrix A is the magnitude of its largest eigenvalue. Prove that $\rho(A) \le \|A\|_2$. [NLA Ex 3.2]

3. Use the in-class worksheet on the following $m \times m$ bidiagonal matrix to answer the below.

$$A = \begin{bmatrix} 1 & 2 & & \\ & 1 & 2 & \\ & & 1 & \ddots \\ & & & \ddots \end{bmatrix}$$

   (a) find a nontrivial lower bound on the condition number $\kappa(A)$

   (b) predict the smallest m such that roughly all significant digits will be lost in the solution $\mathbf{x}$ to a linear system $A\mathbf{x} = \mathbf{b}$ in double precision.

   (c) demonstrate your last claim in a couple of lines of code, by starting with a known $\mathbf{x}$, computing $\mathbf{b}$ then solving via `mldivide`. [Hint: look up the `toeplitz` command to construct the matrix rather than use a loop. You need to choose a $\mathbf{b}$ that causes floating-point rather than exact integer arithmetic to be used!]

4. How many nested loops are implied by each of the following MATLAB commands? (*i.e.* how many loops would you need to write to code the equivalent in C or fortran?) `A = rand(100,100); x = 1:0.1:100; b = A*x; B = A*A;`

5. Give an exact formula, in terms of β, t and m, for the smallest positive integer n that does not belong to the floating-point system $\mathbf{F}$, and compute n for IEEE single- and double-precision. Give one line of code, and its output, which demonstrates this is indeed the case for double-precision. [NLA Ex 13.2]

6. Measure how the time to compute the singular values of a random real dense $m \times m$ matrix scales with m, focusing on the range $10^2 \le m \le 10^3$. Produce a log-log graph of time vs m, and the simple power law to which it is asymptotic. BONUS: for what large m would you expect this to break down and why?

7. Consider the polynomial $p(x) = (x-2)^9 = x^9 - 18x^8 + 144x^7 - 672x^6 + 2016x^5 - 4032x^4 + 5376x^3 - 4608x^2 + 2304x - 512$. [NLA Ex 13.3]

   (a) plot $p(x)$ for $x = 1.920, 1.921, 1.922, \ldots, 2.080$ evaluating p via its coefficients $1, -18, 144, \ldots$.

   (b) overlay on your plot the same computed using $(x-2)^9$

   (c) explain! (including the size of the effect)