Dartmouth College

Mathematics 115 — Practice Problems 1

1. Let $V = M_n(F)$ viewed as a quadratic space of dimension n^2 over F with bilinear form $B(X, Y) = \mathrm{tr}(XY)$, where tr is the trace of the $n \times n$ matrix. Find an orthogonal basis for (V, B) to conclude that $V \cong n\langle 1\rangle \perp m\mathbb{H}$ where $m = (n(n-1))/2$.

2. Let $V = M_n(F)$ viewed as a quadratic space of dimension n^2 over F with bilinear form $B'(X, Y) = \mathrm{tr}(XY^t)$, where tr is the trace of the $n \times n$ matrix, and Y^t the transpose of Y. Find an orthogonal basis for (V, B') to conclude that $V \cong n^2\langle 1\rangle$.

3. Let $\alpha, \beta \in \dot{F} = F^\times$ and (V, B) a regular quadratic space. Show that

$$-\beta \to V \perp \langle\alpha\rangle \iff -\alpha \to V \perp \langle\beta\rangle.$$

4. Let $\alpha, \beta \in F$ with $\alpha^2 + \beta^2 = \gamma \neq 0$. Show that the four-dimensional form $V = \langle 1, 1, -\gamma, -\gamma\rangle$ is hyperbolic (i.e., $V \cong \mathbb{H} \perp \mathbb{H}$.)

5. Show that the following are equivalent:
   (a) Every 4-dimensional quadratic space over F with discriminant -1 is isotropic.
   (b) Every even dimensional quadratic space over F with discriminant -1 is isotropic.
   (c) Every 3-dimensional quadratic space V over F represents its own discriminant.
   (d) Every odd dimensional quadratic space V over F represents its own discriminant.