Homework Assignment #4
Due Wednesday, February 17th

INSTRUCTIONS: As usual, for the "true/false" questions, just circle the correct answer. No justifications are required, but don't guess. You score is based on #right minus #wrong.

1. **TRUE or FALSE**: If X is a normed vector space and if φ is a weakly-continuous linear functional on X, then φ is norm continuous. (Recall that the weak topology on X is the weak topology induced by the pairing with X^*.)

2. **TRUE or FALSE**: If X is a normed vector space and if φ is a norm continuous linear functional on X, then φ is weakly continuous.

3. **TRUE or FALSE**: If X is a normed vector space and ψ is a functional on X^* which is continuous in the weak-$*$ topology, then ψ is norm continuous.

4. **TRUE or FALSE**: If X is a normed vector space and ψ is a functional on X^* which is continuous in the norm topology, then ψ is weak-$*$ continuous.

5. **TRUE or FALSE**: If X is a reflexive Banach space, then the closed unit ball in X is weakly compact.

6. **TRUE or FALSE**: Let X be a normed vector space and let Y be a dense subspace of X^*. Then the weak topology on X induced by the linear functionals in Y coincides with the weak topology on X (induced by all the functionals in X^*).

7. Recall that a set C in a vector space X is called convex if $x, y \in C$ and $\lambda \in [0, 1]$ implies that $\lambda x + (1 - \lambda)y \in C$.

(a) Suppose that $x_1, \ldots, x_n \in X$. If $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, then $\sum_{i=1}^n \lambda_i x_i$ is called a convex combination of the x_i. Show if C is convex, then any convex combination of elements from C belongs to C.

(b) Show that if C is a convex subset of a topological vector space X, then its closure, $\overline{C}$ is also convex.

(c) Work problem E 2.4.1 in the text.

8. Let $\{ F_j \}_{j \in J}$ be a collection of nonempty closed subsets in a compact space X which is totally ordered by reverse containment.[1] Then

$$\bigcap_{j \in J} F_j \neq \emptyset.$$

9. Suppose that X is a compact topological space and that $f : X \to \mathbf{R}$ is continuous. Show that f attains its maximum and minimum on X; that is, show that there are points $y, z \in X$ such that

$$f(y) \leq f(x) \leq f(z) \quad \text{for all } x \in X.$$

(Hint: use Theorem 1.6.2(v).)

10. Work E 2.4.5.

ANS: Let $\iota : X \to X^{**}$ and $\jmath : X^* \to X^{***}$ be the canonical maps. If $B = \{ x \in X : \|x\| \leq 1 \}$, we want to show that $\iota(X)$ is dense in $B^{**} := \{ b \in X^{**} : \|b\| \leq 1 \}$, where B^{**} is given the weak-$*$ topology induced by X^*. Let C be the weak-$*$ closure of $\iota(B)$ in X^{**}. Suppose the contrary that C is not dense in B^{**}. Then there exists an $a \in B^{**} \setminus C$. Since the weak-$*$ topology is locally convex, there is a weak-$*$ open convex neighborhood A of a disjoint from C. Since C is certainly convex, we can apply Theorem 45 from lecture (or 2.4.7) in the book, to find a linear functional ψ, *continuous in the weak-$*$ topology*, and a $t \in \mathbf{R}$, such that

$$\operatorname{Re} \psi(a) < t \leq \operatorname{Re} \Psi(y) \quad \text{for all } y \in C. \tag{1}$$

Since ψ is continuous in the weak-$*$ topology, ψ must be for the form $\jmath(\varphi)$ for some $\varphi \in X^*$.[2] Thus we can rewrite (1) as

$$\operatorname{Re} a(\varphi) < t \leq \operatorname{Re} y(\varphi) \quad \text{for all } y \in C.$$

Since $\iota(B) \subset C$, we have, in particular,

$$\operatorname{Re} a(\varphi) < t \leq \operatorname{Re} \varphi(x) \quad \text{for all } x \in B. \tag{2}$$

Now (2) implies that $\operatorname{Re} a(\varphi) < \|\varphi\|$. But this implies $|a(\varphi)| > \|\varphi\|$ which contradicts the fact that a has norm at most one. This completes the proof.

11. Work E 2.4.6.

ANS: Let $\epsilon > 0$ and $m \in \mathbf{N}$ be given. Let $\overline{C}$ be the norm closure in X of the convex hull $C := \operatorname{conv}\bigl(\{x_n\}_{n=m}^{\infty}\bigr)$. Then $\overline{C}$ is convex by problem 7(b), and therefore weakly closed by Proposition 48 — which says that norm closed convex sets are weakly closed. We are given that $x_n \to x$ weakly. Therefore $x \in \overline{C}$.

[1] Recall that "ordered by reverse containment" simply means that $F_j \geq F_{j'}$ if and only if $F_j \subset F_{j'}$.

[2] This is important. The weak-$*$ topology is what we called the $\tau(X^{**}, X^*)$ topology and is induced by the pairing between X^* and X^{**}. Hence the continuous functionals for this topology are exactly those given my X^*.

Since $\overline{C}$ is the norm closure of C, there is a $y \in C$ such that $\|x - y\| < \epsilon$. But by problem 7c, every $y \in C$ is a convex combination of the x_n with $n \geq m$. Therefore $y = \sum_{n\geq m} \lambda_n x_n$ with $\sum_{n\geq m} \lambda_n = 1$ and only finitely many λ_n non-zero.

12. Suppose that m is a Minkowski functional on a topological vector space X. Let $C := \{\, x \in X : m(x) < 1 \,\}$. Suppose that $\alpha C = C$ for all $\alpha \in \mathbf{F}$ with $|\alpha| = 1$. Show that $m(x) \geq 0$ for all $x \in X$.

ANS: Suppose to the contrary that $m(x) < 0$ for some $x \in X$. Then since $0 = m(0) = m(x + (-x)) \leq m(x) + m(-x)$, we must have $m(-x) > 0$. Let $t > 0$ be such that $tm(-x) = m(-tx) > 1$. Since $m(tx) = tm(x) < 0 < 1$, we have $tx \in C$. Since C is balanced $-tx \in C$. But this implies $m(-tx) < 1$, which contradicts our choice of t.

13. Work E 2.4.16 in the "Revised Printing" of the text. If you don't have access to the revised printing, email me, and I'll send you a pdf of the relevant problem page.[3] (You may want to use question 12.)

ANS: We suppose that m is a Minkowski functional on a vector space X and let

$$C := \{\, x \in X : m(x) < 1 \,\}.^4$$

We say that C is balanced if $\alpha C = C$ for all $\alpha \in \mathbf{F}$ such that $|\alpha| = 1$. We are to show that m is a seminorm if and only if C is balanced.

First, suppose that m is a seminorm. Then if $x \in C$ and if $|\alpha| = 1$, then $m(\alpha x) = |\alpha| m(x) = m(x) < 1$, and $\alpha x \in C$. Thus $\alpha C \subset C$. But then we also have $\bar{\alpha} C \subset C$, and $C = \alpha\bar{\alpha} C \subset \alpha C$. That is, $\alpha C = C$, and C is balanced.

Now suppose that C is balanced. We saw in the previous problem that m maps X into $[0, \infty)$. Since m is subadditive by definition, we just have to establish that m is homogeneous.[5] Let α be such that $|\alpha| = 1$. I claim that $m(\alpha x) = m(x)$ for all $x \in X$. If for some x, $m(x) \neq m(\alpha x)$, then after possibly replacing x by $\bar{\alpha} x$, we may as well assume that $m(\alpha x) < m(x)$. Since $m(\alpha x) \geq 0$, we have $m(x) > 0$. Thus

$$m(\alpha m(x)^{-1} x) = m(x)^{-1} m(\alpha x) < 1.$$

Therefore $\alpha m(x)^{-1} x \in C$. Since $C = \bar{\alpha} C$, we also have $m(x)^{-1} x = \bar{\alpha}\alpha m(x)^{-1} x \in C$. But

$$m\big(m(x)^{-1} x\big) = 1,$$

which contradicts the fact that $m(x) < 1$ for all $x \in C$. This proves the claim.

But each $\alpha \in \mathbf{F}$ can be written as $\alpha = |\alpha| e^{i\theta}$ and we have

$$m(\alpha x) = |\alpha| m(e^{i\theta} x) = |\alpha| m(x).$$

Thus m is homogeneous — and therefore a seminorm.

[3] This problem is key to E 2.4.17 which implies that *every* locally convex topological vector space topology arises from a family of seminorms. You can consider E 2.4.17 an optional "extra-credit" problem. (That means you are welcome to discuss your write-up with me, but don't turn it in.)

[4] The only role the continuity of m plays is to insure that C is open.

[5] Just about everyone's proof was simpler than mine. So I have given the "improved" version.