Homework Assignment #3
Due Wednesday, February 3rd

INSTRUCTIONS: As usual, for the "true/false" questions, just circle the correct answer. No justifications are required, but don't guess. You score is based on #right minus #wrong.

1. **TRUE or FALSE**: The dual of any normed vector space is a Banach space.

2. **TRUE or FALSE**: If X and Y are Banach spaces and $T : X \to Y$ is a surjective linear map, then T is bounded.

   ANS: FALSE: But constructing a counter example is tedious. Let X be any infinite dimensional Banach space. Then X has a basis $\{x_a\}_{a\in A}$ as a vector space over $\mathbf{F}$. (It is a consequence of the Baire Category Theorem that A must be *uncountable*! You might want to prove that for yourself.) Let $\{a_n\}_{n=1}^{\infty}$ be any countable subset of A. We can define a linear functional $\varphi : X \to \mathbf{F}$ simply by arbitrarily specifying what φ does to each x_a. In particular, I can define

$$\Phi(x_a) = \begin{cases} n\|x_{a_n}\| & \text{if } a = a_n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

   Let $y_n = (n\|x_{a_n}\|)^{-1}x_{a_n}$. Then $y_n \to 0$ in X. But $\varphi(y_n) = 1$ for all n. Therefore, $\varphi(y_n) \not\to 0$. Therefore φ is not continuous at 0, and therefore not bounded.

3. **TRUE or FALSE**: If Y is a closed subspace of a normed vector space X and if $x \in X \setminus Y$, then there is a $\varphi \in X^*$ such that $\varphi(y) = 0$ for all $y \in Y$ and $\varphi(x) = 1$.

4. **TRUE or FALSE**: Suppose that X and Y are Banach spaces and that $T_n : X \to Y$ is a bounded linear map for $n = 1, 2, 3, \ldots$. Suppose that there is a linear operator $T_0 : X \to Y$ such that for each $x \in X$, we have $T_n x \to T_0 x$. Then T_0 is bounded.

   ANS: Since $\{T_n\}_{n=1}^{\infty}$ is pointwise bounded, by the Principle of Uniform Boundedness, there is a M such that $\|T_n\| \le M$ for all $n \ge 1$. Now it follows easily that $\|T_0\| \le M$.

5. Suppose that Y is a subspace of a normed vector space X. Show that the closure of Y is given by

$$\overline{Y} = \bigcap \{\ker\varphi : \varphi \in X^* \text{ and } Y \subset \ker\varphi\}.$$

6. Work E.2.3.2 in the text. If may be helpful to think of c_0 as $C_0(\mathbf{N})$. Then if $x \in C_c(\mathbf{N})$, we have $x = \sum x_n \delta_n$, where the x_n are scalars and δ_n is the function taking the value 1 at n and 0 elsewhere.

ANS: This shouldn't be so hard. Recall that $\mathfrak{c}$ and $\mathfrak{c}_0$ are subspaces of ℓ^∞. It is easy to see that $\mathfrak{c}_{00} := C_c(\mathbf{N})$ can be viewed as a dense subspace of either $\mathfrak{c}_0$ or ℓ^1.

Furthermore, if $x \in \ell^\infty$ and $y \in \ell^1$, then

$$\sum_{n=1}^{\infty} |x_n y_n| \le \|x\|_\infty \sum_{n=1}^{\infty} |y_n| = \|x\|_\infty \|y\|_1. \tag{1}$$

Therefore, if $y \in \ell^1$, then we can define $\varphi_y : \mathfrak{c}_0 \to \mathbf{F}$ by

$$\varphi_y(x) = \sum_{n=1}^{\infty} x_n y_n,$$

and $\|\varphi_y\| \le \|y\|_1$. Of course, given $\epsilon > 0$, there is a N such that

$$\sum_{n=1}^{N} |y_n| \ge \|y\|_1 - \epsilon.$$

For $z \in \mathbf{F}$, let $\operatorname{sgn}(z)$ equal $z/|z|$ if $z \ne 0$, and 0 otherwise. (Thus $\overline{\operatorname{sgn}(z)}z = |z|$ for all z.) Define $x \in \mathfrak{c}_0$ by

$$x_n = \begin{cases} \overline{\operatorname{sgn}(y_n)} & \text{if } n \le N, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then x has norm at most one, and

$$\varphi_y(x) = \sum_{n=1}^{N} |y_n| \ge \|y\|_1 - \epsilon.$$

Therefore $\|\varphi_y\| = \|y\|_1$, and $y \mapsto \varphi_y$ is an isometry of ℓ^1 into $\mathfrak{c}_0^*$. (It is obviously linear and one-to-one since it is isometric.) We just have to see that it is surjective.

Suppose that $\varphi \in \mathfrak{c}_0^*$. Define $y_n := \varphi(\delta_n)$. For any N, define

$$x_n^N = \begin{cases} \overline{\operatorname{sgn}(y_n)} & \text{if } n \le N, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then $x^N = \sum_{n=1}^{N} \overline{\operatorname{sgn}(y_n)}\delta_n$, $\|x^N\|_\infty \le 1$ and $x^N \in \mathfrak{c}_{00} \subset \mathfrak{c}_0$. Since

$$\varphi(x^N) = \sum_{n=1}^{N} |y_n| \le \|\varphi\|,$$

$y = (y_n)$ is in ℓ^1. Since $\varphi = \varphi_y$ on $\mathfrak{c}_{00}$, and since $\mathfrak{c}_{00}$ is dense in $\mathfrak{c}_0$, we must have $\varphi = \varphi_y$ as desired. This proves that $\mathfrak{c}_0^*$ is (isometrically isomorphic to) ℓ^1.

Now start with $x \in \ell^\infty$. Then (1) implies that we get a functional $\psi_x : \ell^1 \to \mathbf{F}$ defined by

$$\psi_x(y) = \sum_{n=1}^{\infty} x_n y_n,$$

and that $\|\psi_x\| \le \|x\|_\infty$. If $x \in \ell^\infty$ and if $\epsilon > 0$, then there is a k such that $|x_k| \ge \|x\|_\infty - \epsilon$. Since $\|\delta_k\|_1 = 1$ and since $|\psi_x(\delta_k)| \le \|x\|_\infty - \epsilon$, we see that $x \mapsto \psi_x$ is an isometry of ℓ^∞ into ${\ell^1}^*$. To see that this map is surjective, we proceed as above.[1] Given $\psi \in {\ell^1}^*$, let $x_n := \psi(\delta_n)$. Since $|x_n| \le \|\psi\|$, $x = (x_n) \in \ell^\infty$. Since $\psi = \psi_x$ on $\mathfrak{c}_{00}$ and since $\mathfrak{c}_{00}$ is dense in ℓ^1, we've shown that $\psi = \psi_x$ and that ${\ell^1}^*$ is (isometrically isomorphic to) ℓ^∞.

Now let's look at $\mathfrak{c}^*$. Define $\lambda : \mathfrak{c} \to \mathbf{F}$ by $\lambda(x) = \lim_n x_n$. Then $\lambda \in \mathfrak{c}^*$ and $\|\lambda\| = 1$. Now suppose that $\varphi \in \mathfrak{c}^*$. Then the restriction of φ to $\mathfrak{c}_0 \subset \mathfrak{c}$ is, by the first part of this problem, given by φ_y for some $y \in \ell^1$. On the other hand, if $x \in \mathfrak{c}$, then $x - \lambda(x) \cdot 1 \in \mathfrak{c}_0$, where 1 denotes the constant sequence. If $L := \varphi(1)$, then

$$\varphi(x) = \varphi_y(x) + \lambda(x)(L - \sum y_n).$$

Thus every $\varphi \in \mathfrak{c}^*$ is of the form

$$\varphi(x) = \varphi_y(x) + z\lambda(x)$$

for some $y \in \ell^1$ and $z \in \mathbf{F}$. Furthermore, a straightforward computation shows that $\|\varphi\| = \|y\|_1 + |z|$. Thus we get an isometric isomorphism of $\mathbf{C} \oplus \ell^1$ onto $\mathfrak{c}^*$ where the norm of the latter is given by $\|(z, y)\| := |z| + \|y\|_1$. However it is easy to see that $\mathbf{C} \oplus \ell^1$ is isometrically isomorphic to ℓ^1: just send $(z, (y_n))$ to $(z, y_1, y_2, \dots)$.

Finally, $\mathfrak{c}_0$, and therefore $\mathfrak{c}$, can't be reflexive since $\mathfrak{c}_0$ is separable and $\mathfrak{c}_0^{**} \cong {\ell^1}^* \cong \ell^\infty$ is not.

7. Work E.2.3.4 in the text.

ANS: Let $\{ \varphi_n \}$ be dense in X^*, and choose $x_n \in X$ such that $\|x_n\| = 1$ and such that $|\varphi_n(x_n)| \ge \frac{1}{2}\|\varphi_n\|$. Let Y be the closed linear span of the x_n. Then Y is separable (since the rational span of the x_n is dense in Y). If $X = Y$, then we're done. Otherwise, our Corollary 2.3.5 implies that there is $\varphi \in X^*$ such that $\|\varphi\| = 1$ and such that $\varphi(y) = 0$ for all $y \in Y$. In particular, $\varphi(x_n) = 0$ for all n. But there is a n such that $\|\varphi - \varphi_n\| < \frac{1}{8}$. In particular, $\|\varphi_n\| \ge \frac{1}{2}$. But then

$$\begin{aligned} |\varphi(x_n)| &= |\varphi_n(x_n) - (\varphi_n(x_n) - \varphi(x_n)| \\ &\ge |\varphi_n(x_n)| - |(\varphi - \varphi_n)(x_n)| \\ &\ge \frac{1}{4} - \frac{1}{8} > 0. \end{aligned}$$

This contradicts the fact that $|\varphi(x_n)| = 0$. Therefore $Y = X$ and we're done.

[1] Since counting measure on $\mathbf{N}$ is σ-finite, we could have appealed to the fact that $L^1(X, \mathcal{M}, \mu)^*$ is $L^\infty(X, \mathcal{M}, \mu)$ whenever the measure space is σ-finite, but that would be overkill.

8. Work E.2.3.5 in the text.

ANS: First some comments. For any Banach space X, $\iota : X \to X^{**}$ is an isometric injection. We say that X is reflexive if ι is surjective. Technically, that is not the same as showing that X and X^{**} are isomorphic. Thus saying that since X reflexive implies that X and X^{**} are isomorphic, we have X^* and X^{***} isomorphic is not quite enough to show that X^* is reflexive.

Anyway, to the problem: Assume first that X is reflexive. To show that X^* is reflexive, we need to show that the canonical injection $\iota_{X^*} : X^* \to X^{***}$ is surjective. To this end, suppose that $\Phi \in X^{***}$. Since the composition of bounded maps is bounded, we can define $\varphi \in X^*$ by

$$\varphi(x) := \Phi\bigl(\iota(x)\bigr).$$

Thus we'll be done once we prove that $\iota_{X^*}(\varphi) = \Phi$. However, since $\iota(x)$ is a typical element of X^{**}, we can compute that

$$\begin{aligned}\iota_{X^*}(\varphi)\bigl(\iota(x)\bigr) &= \iota(x)(\varphi)\\ &= \varphi(x)\\ &= \Phi\bigl(\iota(x)\bigr).\end{aligned}$$

This proves that $\iota_{X^*}(\varphi) = \Phi$, and finishes the first half of the problem.

Now suppose that X^* is reflexive so that, in the notation above, $\iota_{X^*} : X^* \to X^{***}$ is surjective. If X were not reflexive, then since $i(X)$ is an isometric image of X, it is complete and therefore it is a closed proper subspace of X^{**}. Therefore, by Corollary 2.3.5, there is a $\Phi \in X^{***}$ such that $\|\Phi\| = 1$ and such that $\Phi\bigl(\iota(X)\bigr) = \{\,0\,\}$. By assumption, we have $\Phi = \iota_{X^*}(\varphi)$ for some $\varphi \in X^*$. But then for all $x \in X$ we have

$$\begin{aligned}0 &= \Phi\bigl(\iota(x)\bigr)\\ &= \iota_{X^*}(\varphi)\bigl(\iota(x)\bigr)\\ &= \iota(x)(\varphi)\\ &= \varphi(x).\end{aligned}$$

But this is absurd, since this implies $\varphi = 0$ in which case $\Phi = \iota_{X^*}(\varphi)$ is zero. Thus we must have $\iota(X)$ equal to all of X^{**} and X is reflexive.

9. Work E.2.3.7 in the text.

ANS: *One* of the challenges here is to write your thoughts down coherently and to properly justify the manipulations with sums.

If $x \in \ell^1$, then for all $\epsilon > 0$, there is a N such that $n \geq N$ implies

$$\Bigl|\sum_{m=n}^{\infty} x_m\Bigr| \leq \sum_{m=n}^{\infty} |x_m| < \epsilon.$$

Therefore

$$(Tx)_n := \sum_{m=n}^{\infty} x_m$$

defines an element Tx in $\mathfrak{c}_0$. Clearly, $T : \ell^1 \to \mathfrak{c}_0$ is linear. Since

$$|(Tx)_n| \le \sum_{m=n}^{\infty} |x_m| \le \|x\|_1,$$

we certainly have $\|Tx\|_\infty \le \|x\|_1$ and $T \in B(\ell^1, \mathfrak{c}_0)$.

Now we identify ℓ^1 with $\mathfrak{c}_0^*$ and ℓ^∞ with ${\ell^1}^*$ via the maps $y \mapsto \varphi_y$ and $x \mapsto \psi_x$ defined in a previous problem. Now if $x, y \in \ell^1$, we have — using Fubini's Theorem to justify the manipulations with sums —

$$\begin{aligned}
(T^*\varphi_x)(y) &= \varphi_x(Ty) \\
&= \sum_{n=1}^{\infty} x_n (Ty)_n \\
&= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} x_n y_m \\
&= \sum_{\{\,(n,m)\in\mathbf{N}\times\mathbf{N} : m \ge n\,\}} x_n y_m \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{m} x_n y_m \\
&= \sum_{m=1}^{\infty} y_m \Big(\sum_{n=1}^{m} x_n\Big) \\
&= \psi_z(y),
\end{aligned}$$

where $z \in \ell^\infty$ is given by $z_m := \sum_{n=1}^{m} x_n$. Therefore as a map from $\ell^1 \to \ell^\infty$, we have $T^*x = z$.