Math 113 Homework
Due 11 January 2010

INSTRUCTIONS: For the "true/false" questions, just circle the correct answer. No justifications are required, but don't guess. You score is based on #right minus #wrong.

1. **TRUE or FALSE**: Suppose that f_i is a continuous real-valued function for $i = 1, \dots n$ (with $n < \infty$). Then $h := \sup_{1 \le i \le n} f_i$ is continuous.

2. **TRUE or FALSE**: Suppose that f_i is a continuous real-valued function for $i = 1, 2, 3, \dots$. Then $h := \sup_i f_i$ is continuous.

In questions 3 to 7, let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers.

3. **TRUE or FALSE**: If $\{a_n\}_{n=1}^{\infty}$ has a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ converging to a, then

$$\liminf_n a_n \le a \le \limsup_n a_n.$$

4. **TRUE or FALSE**: There is a subsequence, $\{a_{n_k}\}_{k=1}^{\infty}$, such that $\lim_k a_{n_k} = \limsup_n a_n$.

5. **TRUE or FALSE**: If $a_n \le b_n$ for all n, then $\limsup_n a_n \le \liminf_n b_n$.

6. **TRUE or FALSE**: $\limsup_n (a_n + b_n) = \limsup_n a_n + \limsup_n b_n$.

7. **TRUE or FALSE**: $\limsup_n a_n = -\liminf_n -a_n$.

On a separate sheet, provide articulate solutions to the following.

8. Prove Lemma 3 from lecture; that is, show that if $T \in B(V, W)$, then $\|Tv\| \le \|T\|\|v\|$ and that $\|T\| = \inf\{\, \alpha \in \mathbf{R}^+ : \|Tv\| \le \alpha\|v\| \text{ for all } v \in V \,\}$.

9. Show that $C([0, 1])$ is a Banach Algebra with respect to the sup norm.

10. [**Corrected Version**] Suppose that V is a normed vector space and that $B := \overline{B_1(0)} = \{\, v \in V : \|v\| \le 1 \,\}$ is the closed unit ball. Let $T : V \to V$ be an operator on V. Show that T is bounded if and only if $T^{-1}(B)$ is a neighborhood of 0 in V. (Hint: $B_\epsilon(0) = \epsilon \cdot B_1(0)$ and T satisfies $T(\epsilon \cdot v) = \epsilon \cdot T(v)$.)