Dartmouth College

Mathematics 81/111 — Homework 2

1. The following is a very standard result whose proof you can easily find, but of course the value to you is in coming up with your own proof.

   Let A be a commutative ring. Show that the following conditions defining a Noetherian ring are equivalent:

   (a) Every ideal of A is finitely generated.

   (b) (ACC) If $I_1 \subseteq I_2 \subseteq \cdots$ is an ascending chain of ideals in A, then there exists an integer $r \geq 1$ so that $I_r = I_{r+1} = \cdots$.

   (c) Every nonempty collections of ideals in A has a maximal element with respect to inclusion.

2. Let $\mathbb{Q}$ be the rational numbers.

   (a) Let A be the ring $\mathbb{Q} \times \mathbb{Q}$. Determine all the ideals of A, and which among those are maximal.

   (b) In case we have not yet proved it in class, assume that the polynomial ring $\mathbb{Q}[x]$ is a PID. Use the Chinese Remainder Theorem (as needed) to help characterize the structure of the quotient rings $\mathbb{Q}[x]/(x^2 - 3x + 2)$ and $\mathbb{Q}[x]/(x^2 + x + 1)$. In particular, where in the spectrum of 'commutative rings to fields' do these rings lie?

   (c) Find the maximal ideas of the rings $\mathbb{Q}[x]/(x^2 - 3x + 2)$ and $\mathbb{Q}[x]/(x^2 + x + 1)$.

3. Consider the ring $A = \mathbb{Z}[\alpha]$ where $\alpha = \sqrt[3]{2}$. Let $I = (5, \alpha + 2)$ and $J = (5, \alpha^2 + 3\alpha - 1)$ be two ideals of A.

   (a) Show that the principal ideal $5A = IJ$.

   (b) Show that there is a surjective ring homomorphism:
   $\mathbb{Z}[x]/(5, x^2 + 3x - 1) \to \mathbb{Z}[\alpha]/(5, \alpha^2 + 3\alpha - 1)$.

   (c) Use the previous part to show that $(5, \alpha^2 + 3\alpha - 1)$ is a maximal ideal in $A = \mathbb{Z}[\alpha]$.

4. Let A be a commutative ring with identity.

   (a) Show that if the polynomial ring $A[x]$ is a PID, then A is a field (hence $A[x]$ is a Euclidean domain).

   (b) Suppose that A is an integral domain containing an irreducible element π. Show that $A[x]$ is not a PID. Hint: Consider the ideal $\langle x, \pi \rangle$.

   (c) Hilbert's Basis theorem says that if A is a Noetherian commutative ring with identity, then so is the polynomial ring $A[x]$. Determine whether the converse is true.

(problems continue on next page)

5. Let $\varphi : \mathbb{C}[x, y] \to \mathbb{C}[t]$ be the ring homomorphism between polynomial rings induced by sending $x \mapsto t^2$ and $y \mapsto t^3$. Show that the kernel of φ is the ideal $(y^2 - x^3)$ and that the image of φ is the subring

$$\{p(t) \in \mathbb{C}[t] \mid p'(0) = 0\},$$

where $p'(t)$ is the first derivative of the polynomial $p(t)$.