Your Name Goes Here

Dartmouth College

Mathematics 81/111 — Homework 5

1. Let L/K be an extension of fields. We say that K **is algebraically closed in** L if the only elements of L which are algebraic over K are the elements of K. Let x be transcendental over K.

   (a) Show that K is algebraically closed in $K(x)$.

   (b) Let ξ be any element of $K(x) \setminus K$. Show that $K(x)/K(\xi)$ is an algebraic extension.

2. Let K/F be a field extension of degree n.

   (a) For any $\alpha \in K$, show that $T_\alpha : K \to K$ given by left multiplication by α is an F-linear transformation on K.

   (b) Prove that K is isomorphic to a subfield of $M_n(F)$. In particular this shows that $M_n(F)$ contains an isomorphic copy of all field extensions of F having degree dividing n.

3. Finding minimal polynomials.

   (a) Let L/F be an algebraic extension of fields, and $\alpha \in L$. Let $m_{\alpha,F} = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be the minimal polynomial of α over F, and put $K = F(\alpha)$. Consider the linear transformation T_α of the previous problem. Compute the matrix $[T_\alpha]_\mathcal{B}$ of T_α with respect to the basis $\mathcal{B} = \{1, \alpha, \ldots, \alpha^{n-1}\}$ of K/F, and show its characteristic polynomial is $m_{\alpha,F}$.

   (b) First, let's do a simple example. Let $F = \mathbb{Q}$ and $\alpha = \sqrt[5]{2}$. Show that the characteristic polynomial of $[T_\alpha]_\mathcal{B}$ is the expected $x^5 - 2$.

   (c) Now suppose that $\beta = 1 + \sqrt[5]{8} + \sqrt[5]{16}$. We know from a simple argument on degrees of towers that the degree of β over $\mathbb{Q}$ is 5. Computing the matrix of T_β with respect to $\{1, \beta, \ldots, \beta^4\}$ would be painful. Instead, compute the matrix of T_β with respect to the basis $\mathcal{B} = \{1, \alpha, \ldots, \alpha^4\}$ where $\alpha = \sqrt[5]{2}$ and the resulting characteristic polynomial. Is this $m_{\beta,\mathbb{Q}}$? If so why? If not, why not?

4. Fields of characteristic p.

   (a) Let K be a field of characteristic p, and let $a \in K$. Show that if a has no pth root in K, then $x^{p^n} - a$ is irreducible in $K[x]$ for any $n \geq 1$.

   (b) Show that every element of a finite field can be written as the sum of two squares in that field.

5. A field K is called **perfect** if either it has characteristic 0, or has characteristic p and $K^p = K$ (that is, the Frobenius map is an automorphism of K).
   (a) Show that any algebraic extension of a perfect field is separable.
   (b) Give (and justify) an example of a field which is not perfect.

6. Let $\sqrt[7]{2}$ be the real seventh root of 2, and ζ_7 a primitive 7th root of unity in $\mathbb{C}$. Show that -1 cannot be written as a sum of squares in $\mathbb{Q}(\sqrt[7]{2}\zeta_7)$