Dartmouth College

Mathematics 81/111 — Homework 3

1. A first approximation to a theorem of Bézout. It shows that if two curves in the plane are described by the zero sets of relatively prime polynomials, then the two curves intersect in only a finite number of points. For example in $\mathbb{R}^2$ we expect the parabola $Z(y-x^2)$ will intersect a circle $Z(x^2+(y-a)^2-4)$ in anywhere from 0-4 points, depending on the value of a. Bézout's theorem gives as an upper bound the product of the degrees of the polynomials with equality holding in a nice setting like the projective plane $\mathbb{P}^2(\mathbb{C})$. In this problem, you show only that the number of points of intersection is finite.
   (a) Let A be a UFD with field of fractions K. Let $f, g \in A[x]$. Show that f, g are relatively prime in $A[x]$ if and only if f, g are relatively prime in $K[x]$ and their contents, $C(f)$ and $C(g)$, are relatively prime in A.
   (b) Let k be a field and $f, g \in k[x, y]$ be relatively prime. Show that $Z(\{f, g\}) = Z(f) \cap Z(g)$ is a finite set of points, that is show that the algebraic set consisting of the common zeros of f and g is finite.
2. Let F be a field and $f \in F[x]$ a non-constant polynomial.
   (a) Characterize all the maximal ideals in $F[x]$ containing $\langle f \rangle$.
   (b) Determine all the ideals of the quotient ring $\mathbb{Z}[x]/\langle 5, x^3+2 \rangle$.
3. Let $n \geq 1$ and consider the polynomial $f(x, y) = y^n - (x^3 - x) \in \mathbb{Q}[x, y]$.
   (a) Characterize the quotient $\mathbb{Q}[x, y]/(y, f(x, y))$ in terms of familiar rings.
   (b) Show that $A = \mathbb{Q}[x, y]/(f)$ is an integral domain containing an isomorphic copy of $\mathbb{Q}[x]$. Show moreover that A is finitely generated $\mathbb{Q}[x]$-module, in particular, that $A \subseteq \mathbb{Q}[x](1 + (f)) + \mathbb{Q}[x](y + (f)) + \cdots + \mathbb{Q}[x](y^{n-1} + (f))$.
   (c) Show that $B = \mathbb{Q}[x, y]/(x, f(x, y))$ is an integral domain if and only if $n = 1$, but in any case is a finite dimensional vector space over $\mathbb{Q}$.
4. Let A be a commutative ring with identity having prime characteristic p, and let $\varphi : A \to A$ be the map $\varphi(a) = a^p$. The map φ is called the *Frobenius map*.
   (a) Show that φ is a ring homomorphism.
   (b) Henceforth assume that the ring A is specialized to a field F (having characteristic p). Show that $\varphi : F \to F$ is injective.
   (c) Show that if F is a finite field, then φ is surjective.
   (d) Show that if F is a finite field, then for any $f \in F[x]$, there exists a $g \in F[x]$ with $f(x^p) = (g(x))^p$.
   (e) Show that if $F = \mathbb{Z}/p\mathbb{Z}$, then φ is the identity map, and that for any $f \in F[x]$, $f(x^p) = (f(x))^p$.

(continued on next page)

5. Hint: If you don't know where to begin, I would suggest looking up diagonalizable in Lang's index. The reference will give (as a homework problem) an important theorem.
   (a) Let A be an $n \times n$ matrix over the complex numbers, $\mathbb{C}$, for which $A^k = I_n$ is the identity matrix for some integer $k \geq 1$. Show that A is diagonalizable.
   (b) Let F be a field of prime characteristic p, and $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \in M_2(F)$. Show that $A^p = I_2$, and that A is diagonalizable if and only if $\alpha = 0$.

6. For a commutative ring A, define the *Krull dimension* of A to be the maximum possible length of a chain $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ of distinct prime ideals in A. The given chain has length n and there are rings with infinite dimension.
   (a) Show that an integral domain with Krull dimension 0 is a field.
   (b) Show that every PID (which is not a field) has Krull dimension 1.
   (c) Show that $\mathbb{Z}[x_1, \ldots, x_n]$ has Krull dimension $\geq n + 1$.