Dartmouth College

Mathematics 81/111 — Homework 2

1. Let $\mathbb{Q}$ be the rational numbers.
   (a) Let A be the ring $\mathbb{Q} \times \mathbb{Q}$. Determine all the ideals of A, and which among those are maximal.
   (b) In case we have not yet proved it in class, assume that the polynomial ring $\mathbb{Q}[x]$ is a PID. Use the Chinese Remainder Theorem (as needed) to help characterize the structure of the quotient rings $\mathbb{Q}[x]/(x^2 - 3x + 2)$ and $\mathbb{Q}[x]/(x^2 + x + 1)$. In particular, where in the spectrum of 'commutative rings to fields' do these rings lie?
   (c) Find the maximal ideas of the rings $\mathbb{Q}[x]/(x^2 - 3x + 2)$ and $\mathbb{Q}[x]/(x^2 + x + 1)$.

2. Let $\varphi : \mathbb{C}[x, y] \to \mathbb{C}[t]$ be the ring homomorphism between polynomial rings induced by sending $x \mapsto t^2$ and $y \mapsto t^3$. Show that the kernel of φ is the ideal $(y^2 - x^3)$ and that the image of φ is the subring

$$\{p(t) \in \mathbb{C}[t] \mid p'(0) = 0\},$$

where $p'(t)$ is the first derivative of the polynomial $p(t)$.

In the next couple of exercises, we explore the notions of prime and maximal ideals in non-commutative rings. We start with two basic definitions. Here A is a general ring which does not necessarily have an identity.

- *Definition:* A ring A is **simple** if $A^2 \neq 0$, and its only (2-sided) ideals are $\{0\}$ and A.
- *Definition:* Let P be an ideal in a ring A. We say that P is a **prime ideal** if $P \neq A$ and if for any ideals $I, J \subseteq A$, $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

3. (a) Let k be a field, and $n \geq 1$ an integer. Show that the matrix ring $M_n(k)$ is a simple ring. *Hint:* Let $E_{i,j}$ be the $n \times n$ matrix with a 1_k in the i, j-entry, and 0_k elsewhere. For a matrix $A \in M_n(k)$, it might be interesting to compute $E_{p,r}AE_{s,q}$.
   (b) Let A be a non-trivial ring with identity, $P \subseteq A$ an ideal, $P \neq A$.
   - Show that if for all $a, b \in A$, $ab \in P$ implies $a \in P$ or $b \in P$, then P is a prime ideal (in the sense defined above).
   - Show that if A is commutative, and P is prime in the above sense, then for all $a, b \in A$, we have $ab \in P$ implies $a \in P$ or $b \in P$. Where did you use commutativity?
   - Find an example of an ideal in a non-commutative ring which is prime, but fails to satisfy the prime definition for commutative rings.

(c) Let A be a ring with identity. Show that an ideal $M \subseteq A$ is maximal if and only if A/M is simple and nontrivial. Show that every maximal ideal is prime.

4. Let $\varphi : A \to B$ be a surjective homomorphism of (not necessarily commutative) rings with identity.

   (a) Let $Q \subset B$ be a prime ideal. Show that $\varphi^{-1}(Q)$ is a prime ideal in A.

   (b) Let $Q \subset B$ be a maximal ideal. Show that $\varphi^{-1}(Q)$ is a maximal ideal in A. Give an example where φ is not surjective, $\varphi^{-1}(Q) \neq A$, and yet $\varphi^{-1}(Q)$ is not maximal.

   (c) Show that if I is an ideal of A, then $\varphi(I) = B$ if and only if $I + \ker(\varphi) = A$. Conclude that if M is a maximal ideal in A, that $\varphi(M)$ is maximal if and only if $\ker(\varphi) \subseteq M$.

5. Let A be a commutative ring with identity.

   (a) Show that if the polynomial ring $A[x]$ is a PID, then A is a field (hence $A[x]$ is a Euclidean domain).

   (b) Suppose that A is an integral domain containing an irreducible element π. Show that $A[x]$ is not a PID. Hint: Consider the ideal $\langle x, \pi \rangle$.

   (c) Hilbert's Basis theorem says that if A is a Noetherian commutative ring with identity, then so is the polynomial ring $A[x]$. Determine whether the converse is true.