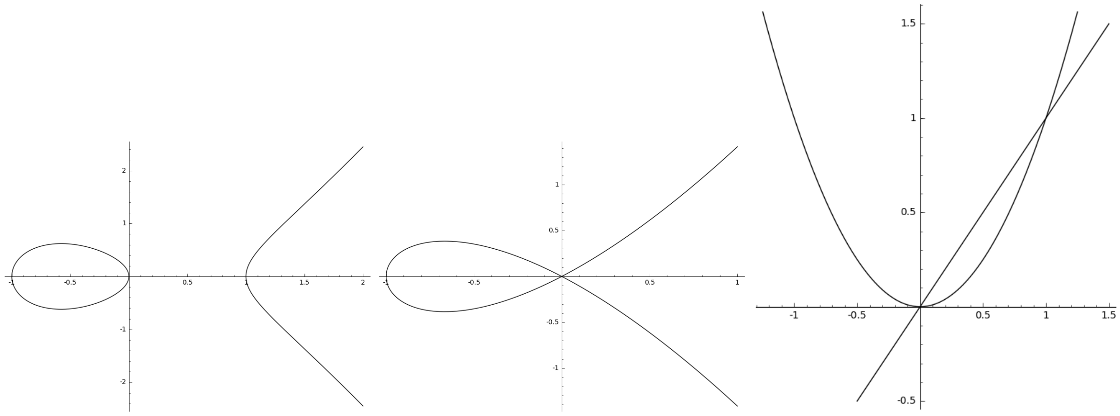
Dartmouth College
Mathematics 81/111 — Homework 1

Some basic definitions concerning algebraic sets

Let k be a field, and let $k[x_1, \ldots, x_n]$ the polynomial ring in n variables with coefficients in k. For $f \in k[x_1, \ldots, x_n]$ and $P = (a_1, \ldots, a_n) \in k^n$, we write $f(P)$ for $f(a_1, \ldots, a_n)$, and define the **zero set** of f by

$$Z(f) = \{P \in k^n \mid f(P) = 0\}.$$

For example, consider three real plane curves $Z(f)$, i.e., where $f \in \mathbb{R}[x, y]$.



(a) $Z(y^2 - x(x^2 - 1))$ (b) $Z(y^2 - x^2(x + 1))$ (c) $Z(y^2 - xy - x^2y + x^3)$

1. For a subset $S \subseteq k[x_1, \ldots, x_n]$, define the **zero set of** S to be the common zeros of all the elements of S, that is

$$Z(S) = \{P \in k^n \mid f(P) = 0 \text{ for all } f \in S\} = \bigcap_{f \in S} Z(f).$$

2. A subset $X \subseteq k^n$ is called an (affine) **algebraic set** if $X = Z(S)$ for some $S \subseteq k[x_1, \ldots, x_n]$. Let $\mathcal{A}$ denote the set of all affine algebraic subsets of k^n. It is trivial to check that $Z(S) = Z(\,(S)\,)$, where (S) is the ideal generated by the set S in $k[x_1, \ldots, x_n]$.

3. Let $\mathcal{I}$ denote the set of ideals in $k[x_1, \ldots, x_n]$, so every element of $\mathcal{A}$ is of the form $Z(J)$ for some $J \in \mathcal{I}$.

4. For any subset $Y \subseteq k^n$ define the **ideal of** Y, $I(Y) \in \mathcal{I}$, by

$$I(Y) = \{f \in k[x_1, \ldots, x_n] \mid f(P) = 0 \text{ for all } P \in Y\}.$$

We have now defined two functions: $Z : \mathcal{I} \to \mathcal{A}$ which maps subsets of $k[x_1, \ldots, x_n]$ to affine algebraic sets, and a function $I : \mathcal{A} \to \mathcal{I}$, which maps subsets of k^n to ideals.

The Exercises

In the first few exercises, we establish some basic properties of the maps defined above.

1. Show that the union of two algebraic sets is an algebraic set. Show that the intersection of any family of algebraic sets is an algebraic set. Show that the empty set and the whole space (k^n) are algebraic sets.

   If we take the algebraic sets as the closed sets in k^n, the properties above for closed sets defines a topology on k^n, called the *Zariski topology*.

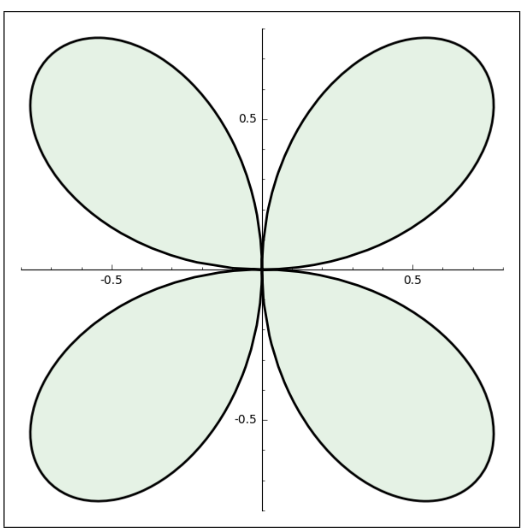
2. We shall take as given that $I(\emptyset) = k[x_1, \ldots, x_n]$. Show that $I(k^n) = \{0\}$ if and only if k is an infinite field. Also show that if $P = (a_1, \ldots, a_n) \in k^n$, then $I(P) = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$, the ideal generated by $\{x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n\}$.

3. Some basic correspondences:
   (a) Show that $I(Z(S)) \supseteq S$ for any set $S \subseteq k[x_1, \ldots, x_n]$, and $Z(I(X)) \supseteq X$ for any set $X \subseteq k^n$. **Hint:** First establish the very useful observation that both functions I and Z are inclusion reversing.
   (b) Deduce that $Z(I(Z(S))) = Z(S)$ for any set $S \subseteq k[x_1, \ldots, x_n]$, and $I(Z(I(X))) = I(X)$ for any set $X \subseteq k^n$, that is $Z(I(V)) = V$ for any algebraic set V, and $I(Z(J)) = J$ for any J which is the ideal of a set in k^n.
   (c) Show that for any set $Y \subseteq k^n$, we have $Z(I(Y)) = \overline{Y}$, the closure of Y in the Zariski topology.

   Remark. If k is algebraically closed (e.g., $k = \mathbb{C}$), then for any ideal $J \subseteq k[x_1, \ldots, x_n]$, Hilbert's Nullstellensatz shows that $I(Z(J)) = \sqrt{J}$ where

   $$\sqrt{J} = \{f \in k[x_1, \ldots, x_n] \mid f^r \in J \text{ for some } r \in \mathbb{Z}_+\}, \text{ is called the } \textbf{radical} \text{ of } J.$$

4. Consider the four-leaved rose given as the polar plot of $r = \sin(2\theta)$. We wish to show that this curve is an algebraic set.



Start with the usual change of coordinates from polar to Cartesian coordinates:

$$x = r\cos\theta; \qquad y = r\sin\theta; \qquad (x^2 + y^2 = r^2).$$

First show that every point of the polar curve is on the algebraic set $Z((x^2 + y^2)^3 - 4x^2y^2))$; a simple trig identity may be useful. Then show the converse. The converse requires a bit of thought (and explanation).

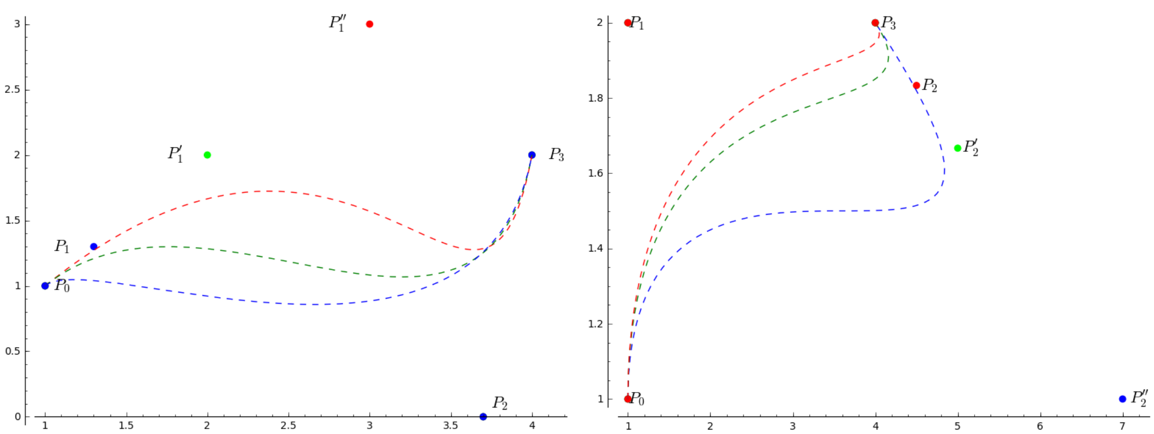
5. Let k be an infinite field, Y be the algebraic set $Z(y-x) \subset k^2$, and let $X = Y \setminus \{(1,1)\}$, that is, the line minus the point $(1,1)$. We want to show that X is not an algebraic set. Proceed by contradiction, assuming $X = Z(S)$ for some set of polynomials. Since $Z(S) = Z((S))$, we may assume (by the Hilbert Basis Theorem (soon to come)) that $S = \{f_1, \ldots, f_n\}$ is a finite set. Here each $f_i \in k[x,y]$.
   (a) Show that for any $f \in k[x,y]$, $f \in I(X)$ implies $f \in I(Y)$.
   (b) Use previous problems in this homework set to obtain a contradiction.

6. In the last two problems, we consider cubic Berstein polynomials, and their associated plane curves, called Bézier curves. While Berstein's work was involved with approximation theory, they were used in 1959 by Casteljau to help design car bodies for Citroën. That use was better publicized (1962) by Bézier, an automobile designer for Renault, and so now has his name attached to them. They are also fundamental in the PostScript language and play a crucial role in Knuth's METAFONT system.

   The idea is quite simple. You want to produce a curve knowing the starting and ending points, as well as the tangent directions at the endpoints. To do this, we need four (control) points: $\mathbf{P}_0, \ldots, \mathbf{P}_3$ with $\mathbf{P}_j$ having coordinates (x_j, y_j). Points on the cubic Bézier curve are given parametrically by:

$$\mathbf{P}(t) = (x(t), y(t)) = (1-t)^3\mathbf{P}_0 + 3t(1-t)^2\mathbf{P}_1 + 3t^2(1-t)\mathbf{P}_2 + t^3\mathbf{P}_3, \quad 0 \le t \le 1.$$

   (a) Trivially, we see that $\mathbf{P}(0) = \mathbf{P}_0$ and $\mathbf{P}(1) = \mathbf{P}_3$. Show that the tangent directions at the endpoints are parallel to the vectors from $\mathbf{P}_0$ to $\mathbf{P}_1$, and from $\mathbf{P}_2$ to $\mathbf{P}_3$. Thus positioning the control points $\mathbf{P}_1$ and $\mathbf{P}_2$ allows a great deal of flexibility in the shape of the curve.
   (b) Explain what happens if you replace $\mathbf{P}_1$ by a point $\mathbf{P}_1'$ which lies on the line through $\mathbf{P}_0$ and $\mathbf{P}_1$, but which is nearer or farther away from $\mathbf{P}_0$ than $\mathbf{P}_1$. As a visual cue, see the pictures below which show the effect of moving one of the two control points P_1 or P_2.



7. Next we want to show that a Bézier curve always lies within the convex hull (control polygon) determined by the four control points.

(a) Recall that a subset $C \subseteq \mathbb{R}^2$ is call *convex* if for points $\mathbf{P}, \mathbf{Q} \in C$ the line segment joining $\mathbf{P}$ to $\mathbf{Q}$ lies in C. Show that if $\mathbf{P}, \mathbf{Q} \in C$, then all points $t\mathbf{P} + (1-t)\mathbf{Q} \in C$, for $0 \leq t \leq 1$.

(b) Show that if $\mathbf{P}_1, \ldots, \mathbf{P}_n$ lie in a convex subset C, then so does $\sum_{i=1}^{n} t_i \mathbf{P}_i$ where the t_i are nonnegative real numbers whose sum is 1.

(c) Show that every point on the Bézier curve lies in the convex polygon determined by the four control points which define it. Some samples are pictured below: